REPRESENTATIONS OF FUNCTIONS HARMONIC IN THE UPPER HALF-PLANE AND THEIR APPLICATIONS

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By

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

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ABSTRACT

REPRESENTATIONS OF FUNCTIONS HARMONIC IN THE UPPER HALF-PLANE AND THEIR APPLICATIONS

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September, 2003

In this thesis, new conditions for the validity of a generalized Poisson representation for a function harmonic in the upper half-plane have been found. These conditions differ from known ones by weaker growth restrictions inside the half-plane and stronger restrictions on the behavior on the real axis.

We applied our results in order to obtain some new factorization theorems in Hardy and Nevanlinna classes.

As another application we obtained a criterion of belonging to the Hardy class up to an exponential factor.

Finally, our results allowed us to extend the Titchmarsh convolution theorem to linearly independent measures with unbounded support.

Keywords: Analytic curves, generalized Poisson integral, Hardy class, Nevanlinna class, Nevanlinna characteristics, Titchmarsh convolution theorem.

ÖZET

ÜST YARI DÜZLEMDEKİ HARMONİK FONKSİYONLARIN GÖSTERİMLERİ VE BUNLARIN UYGULAMALARI

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Bu tezde üst yarı düzlemdeki harmonik fonksiyonların genelleştirilmiş Poisson integrali biçiminde gösterilebilmelerinin yeni koşullarını bulduk. Bu koşullar, üst yarı düzlemde daha zayıf büyüme sınırlandırmaları ve reel eksende daha kuvvetli sınırlandırmalarla bilinen koşullardan farklılık gösterirler.

Sonuçlarımızı, Hardy ve Nevanlinna sınıflarına, bu sınıflarda yeni faktorizasyon teoremleri bulmak için uyguladık.

Diğer bir uygulama olarak, fonksiyonların bir üstel çarpanla birlikte Hardy sınıfına ait olmalarının bir kriterini bulduk.

Son olarak, sonuçlarımız Titchmarsh'ın konvolüsyon teoremini sonsuz dayanaklı, lineer bağımlı ölçümlere genişletmemize olanak sağladı.

Anahtar sözcükler: Analitik eğriler, genelleştirilmiş Poisson integrali, Hardy sınıfı, Nevanlinna sınıfı, Nevanlinna karakteristikleri, Titchmarsh'ın konvolüsyon teoremi.

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Contents

Chapter 1

Introduction

It is well-known from Complex Analysis that if a function u is harmonic in the disk $D_R := \{z \in \mathbb{C} : \ |z| < R\}$ and continuous on its closure, then u admits the following Poisson representation

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2} u(Re^{i\theta}) d\theta, \quad z = re^{i\varphi} \in D_R.$$

The counterpart of this representation for the upper half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \ \operatorname{Im} z > 0\}$ is the following:

$$u(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{u(t)}{(x-t)^2 + y^2} dt, \quad z = x + iy \in \mathbb{C}_+. \tag{1.1}$$

Unfortunately, this representation does not hold generally for functions harmonic in $\mathbb{C}_+$, and continuous on its closure $\overline{\mathbb{C}_+} := \{z \in \mathbb{C} : \ \operatorname{Im} z \geq 0\}$, even if the integral on the right hand side converges. For example, take u as the imaginary part of any polynomial with real coefficients. The conditions for the validity of representation (1.1) are rather restrictive; roughly speaking, they are of the kind $u(z) = o(|z|), \ |z| \to \infty$.

The following more general representation

$$u(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\nu(t)}{(x-t)^2 + y^2} + cy, \quad z = x + iy \in \mathbb{C}_+, \tag{1.2}$$

of a harmonic function u in $\mathbb{C}_+$, where ν is a σ-finite Borel measure on $\mathbb{R}$ and c is a constant, is also very important and has applications in the theory of integral transforms [2, Ch.4], in the theory of entire functions [19, Part II], [18, Ch.5], [17, Ch.3], in the theory of H_p spaces [16, Ch.6].

It is well-known (see, e.g. [16, p.107], [19, p.100]) that (1.2) holds if and only if u can be represented in the form $u = u_1 - u_2$ where u_1 and u_2 are non-negative harmonic functions in $\mathbb{C}_+$. Nevertheless, for several applications (see, e.g. [17, Ch.3], [18, Ch.5]) conditions that can be expressed in terms of the *growth* of u are more useful. For a function u continuous on $\overline{\mathbb{C}_+}$ the strongest result of this kind was obtained by R. Nevanlinna [21].

The present thesis is devoted to the conditions of the validity of a more general representation, including (1.2) as a special case, and some of its applications.

This representation has the form

$$u(z) = \int_{-\infty}^{\infty} P_q(z,t)d\nu(t) + \mathrm{Im}P(z), \quad z \in \mathbb{C}_+. \tag{1.3}$$

Here $P_q(z,t)$ is the generalized Poisson kernel defined by the formula

$$P_q(z,t) = \mathrm{Im}\left\{\frac{1}{\pi}\frac{(1+tz)^q}{(t-z)(1+t^2)^q}\right\}, \quad q \in \mathbb{N} \cup \{0\},$$

the measure ν is a σ-finite Borel measure on $\mathbb{R}$, and P is a polynomial of degree at most q. Note that, if we put $q = 1$ in (1.3), we obtain the representation (1.2). The representation (1.3) for u being continuous in $\mathbb{C}_+$ was first considered by R. Nevanlinna [21].

R. Nevanlinna [21] and later N. Govorov [14] showed that representation (1.3) is valid under a growth condition on u in the upper half-plane, and a condition on the behavior of u near the real line. Our results differ from the cited ones by a remarkably relaxed growth condition in the upper half-plane.

We applied our results on representation of harmonic function in upper half-plane to obtain some new factorization theorems in Hardy and Nevanlinna classes.

The classical factorization theorems in Hardy and Nevanlinna classes [7, Ch.11], [15, Ch.8], [16, Ch. VI], are well-known and have plenty of applications in Complex Analysis and Functional Analysis [7, 15, 16, 19]. In 1985, I.V. Ostrovskii [23] proved a factorization theorem in the Hardy class $H^\infty(\mathbb{C}_+)$ of a different kind. This theorem was a basis of his extension [23] of the Titchmarsh convolution theorem to measures with unbounded support.

In this thesis, we applied the theorem on the representation of harmonic functions in the upper half-plane to obtain a factorization theorem which improves and extends the mentioned theorem of [23] in several manners. One of them is an improvement of the theorem in the case when the factors are linearly dependent. The last result is used to get a counterpart of the result of [23] for the linearly dependent measures with unbounded support.

We also applied the representation theorem to obtain a criterion of belonging to the Hardy class $H_p(\mathbb{C}_+)$ up to an exponential factor.

The results of this thesis have been published [8], [9], [11], [12] and accepted [10] for publication.

Chapter 2

Statement of results

2.1 Generalized Poisson representation of a function harmonic in the upper half-plane

In this thesis we found some conditions of the validity of the following representation of a real-valued harmonic function u in $\mathbb{C}_+$:

$$u(z) = \int_{-\infty}^{\infty} P_q(z,t)d\nu(t) + \operatorname{Im} P(z), \quad z \in \mathbb{C}_+, \tag{2.1}$$

where $P_q(z,t)$ is the generalized Poisson kernel defined by the formula

$$P_q(z,t) = \operatorname{Im}\left\{\frac{1}{\pi}\frac{(1+tz)^q}{(t-z)(1+t^2)^q}\right\}, \quad q \in \mathbb{N} \cup \{0\},$$

ν is a σ-finite Borel measure on $\mathbb{R}$ satisfying

$$\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1+|t|^{q+1}} < \infty,$$

and P is a real polynomial of degree at most q.

Further, we assume that all harmonic functions and Borel measures are real-valued.

For functions u harmonic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}_+}$, R. Nevanlinna gives the strongest result on the validity of the representation (2.1):

Theorem A ([21]) *Let u be a function harmonic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}_+}$ and satisfying the conditions:*

(i) *There exists a sequence $\{r_k\}$, $r_k \to \infty$, such that*

$$\int_0^\pi u^+(re^{i\theta}) \sin\theta d\theta = O(r^q), \ r = r_k \to \infty, \tag{2.2}$$

(ii)

$$\int_{-\infty}^\infty \frac{u^+(t)}{1+|t|^{q+1}} dt < \infty. \tag{2.3}$$

Then u admits representation (2.1) *with $d\nu(t) = u(t)dt$.*

In [14], N. V. Govorov showed that the continuity assumption in the closed half- plane can be dropped under some condition:

Theorem B ([14]) *Let u be a function harmonic in $\mathbb{C}_+$, if*

$$\max_{0<\theta<\pi} u^+(re^{i\theta}) = O(r^\alpha), \ \textit{for some } \alpha, \ \alpha < q, \tag{2.4}$$

then u admits representation (2.1).

Our main result is the following:

Theorem 2.1 *Let u be a function harmonic in $\mathbb{C}_+$ and satisfying the following conditions:*

(i) *There exists a sequence $\{r_k\}$, $r_k \to \infty$, such that*

$$\int_0^\pi u^+(re^{i\theta}) \sin\theta d\theta \le \exp\{o(r)\}, \ r = r_k \to \infty. \tag{2.5}$$

(ii) *There exists $\alpha > 0$ such that*

$$\liminf_{s \to 0^+} \int_{-\infty}^\infty \frac{|u(t+is)|}{1+|t|^\alpha} dt < \infty. \tag{2.6}$$

Then u admits representation (2.1) *where* $q = \max\{n \in \mathbb{N} \cup \{0\} : \ n < \alpha\}$, ν *is a* σ*-finite Borel measure on* $\mathbb{R}$ *satisfying*

$$\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1 + |t|^{\alpha}} < \infty,$$

and P is a real polynomial of degree at most q.

Note that Nevanlinna's [21] and Govorov's [14] results, mentioned above, are not contained in Theorem 2.1 because neither (2.3) nor (2.4) imply (2.6), and our result differs from the Nevalinna's and Govorov's results by much weaker growth condition (2.5) on the upper half-plane comparatively with (2.2) and (2.4).

The following immediate corollary of Theorem 2.1, gives the conditions of validity of usual Poisson representation of a function harmonic in $\mathbb{C}_+$.

Corollary 2.2 *Let u be a function harmonic in $\mathbb{C}_+$ and satisfying the following conditions:*

(i) *There exists a sequence* $\{r_k\}$, $r_k \to \infty$, *such that*

$$\int_0^{\pi} u^+(re^{i\theta}) \sin\theta d\theta \leq \exp\{o(r)\}, \ r = r_k \to \infty, \tag{2.7}$$

(ii)

$$\liminf_{s \to 0^+} \int_{-\infty}^{\infty} \frac{|u(t + is)|}{1 + t^2} dt < \infty. \tag{2.8}$$

Then u admits representation

$$u(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\nu(t)}{(x - t)^2 + y^2} + cy, \quad z = x + iy \in \mathbb{C}_+, \tag{2.9}$$

where ν is a σ-finite Borel measure on $\mathbb{R}$ satisfying

$$\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1 + t^2} < \infty,$$

and c is a real constant.

Assumptions (2.5) and (2.6) in Theorem 2.1 are sharp in the following sense: "o" cannot be replaced by "O" in (2.5) as the example $u(z) = \mathrm{Re}\{\cos z\}$ shows. Moreover, (2.6) cannot be replaced by

$$\int_{-\infty}^{\infty} \frac{|u(t+iH)|}{1+|t|^{\alpha}} dt < \infty,$$

for some $H > 0$. This follows from the example $u(z) = \mathrm{Im}\{(z - iH)^{2n}\},\ n > (\alpha - 1)/2,\ n \in \mathbb{N}$. It is also worth mentioning that $|u(t+is)|$ cannot be replaced with $u^{+}(t+is)$ in (2.6) as the example $u(z) = -\mathrm{Re}\{z^{2n}\},\ n > (\alpha-1)/2,\ n \in \mathbb{N}$, shows.

We also considered the possibility of weakening the condition (2.6) in Theorem 2.1. For example, is it possible to replace (2.6) by a condition requiring convergence of the integrals in (2.6) only over two horizontal lines? In some sense, there is an affirmative answer to this question. To formulate our result more precisely, we need a lemma.

Lemma 2.3 *Let $u(z)$ be a function harmonic in $\mathbb{C}_+$ and satisfying the following condition:*

$$\exists H > 0,\ \forall R > 0,\ \sup_{0<y<H} \int_{-R}^{R} |u(x+iy)| dx < \infty. \tag{2.10}$$

Then there exists a Borel measure ν on $\mathbb{R}$ such that for all $R > 0$ it satisfies $|\nu|([-R, R]) < \infty$, and the function

$$u(z) - \int_{-R}^{R} P_q(z,t) d\nu(t), \quad q \in \mathbb{N} \cup \{0\},$$

is harmonic in $\mathbb{C}_+$, continuous in $\mathbb{C}_+ \cup (-R, R)$ and vanishes on $(-R, R)$.

Our next result is the following:

Theorem 2.4 *Let u be a function harmonic in $\mathbb{C}_+$ satisfying condition* (2.10) *of Lemma 2.3 and* (2.5) *of Theorem 2.1. Assume additionally that u satisfies the following condition:*

There exist $H > 0$ and $\alpha > 0$ such that

$$\int_{-\infty}^{\infty} \frac{|u(t+iH)|}{1+|t|^{\alpha}} dt + \int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1+|t|^{\alpha}} < \infty, \tag{2.11}$$

where ν is the σ-finite Borel measure defined in Lemma 2.3.

Then u admits representation (2.1)*, where q and P are as in Theorem 2.1.*

The following corollary to Theorem 2.4 is immediate.

Corollary 2.5 *Let u be a function harmonic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}_+}$ and satisfying* (2.5) *of Theorem 2.1. Suppose there exist $H > 0$ and $\alpha > 0$ such that*

$$\int_{-\infty}^{\infty} \frac{|u(t)| + |u(t+iH)|}{1+|t|^{\alpha}} dt < \infty. \tag{2.12}$$

Then u admits representation (2.1) *with $d\nu(t) = u(t)dt$, where q and P are as in Theorem 2.1.*

In Chapter 5, we will prove a weakened version of Theorem 2.1 where condition (2.6) is replaced by a stronger one, then we will prove Lemma 2.3 and Theorem 2.4. Finally, we will prove Theorem 2.1 using its weakened version and Theorem 2.4.

2.2 Applications to Hardy and Nevanlinna classes

The Hardy class $H^p(\mathbb{C}_+)$, $0 < p \leq \infty$, consists of all functions f analytic in the upper half-plane $\mathbb{C}_+$ and satisfying the condition

$$\sup_{0<y<\infty} \|f(\cdot + iy)\|_p < \infty,$$

where

$$\|h(\cdot)\|_p = \left(\int_{-\infty}^{\infty} |h(x)|^p dx\right)^{\min(1,1/p)}, \quad 0 < p < \infty,$$
$$\|h(\cdot)\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}} |h(x)|.$$

In 1985, I.V. Ostrovskii [23] proved the following factorization theorem in the Hardy class $H^\infty(\mathbb{C}_+)$. This theorem was a basis of his extension [23] of the Titchmarsh convolution theorem to measures with unbounded support.

Theorem C ([23]) *Let $h \not\equiv 0$ belong to $H^\infty(\mathbb{C}_+)$. Assume that $h = g_1 g_2$ where g_1 and g_2 are analytic in $\mathbb{C}_+$ and satisfying the following conditions:*

(i) *There exists a sequence $\{r_k\}$, $r_k \to \infty$, such that*

$$\sup\{|g_1(z)| + |g_2(z)| : \ |z| < r, \ \operatorname{Im} z > 0\} \le \exp\exp\{o(r)\}, \ r = r_k \to \infty. \tag{2.13}$$

(ii) *There exists $H > 0$ such that*

$$\sup\{|g_1(z)| + |g_2(z)| : \ 0 < \operatorname{Im} z < H\} < \infty.$$

Then there exist real constants α_1, α_2 such that

$$g_j(z) e^{i\alpha_j z} \in H^\infty(\mathbb{C}_+), \ j = 1, 2.$$

It is possible to extend Theorem C to classes wider than $H^\infty(\mathbb{C}_+)$. To be more precise, recall that the *Nevanlinna class* is the set of all functions f analytic in $\mathbb{C}_+$ such that $\log|f|$ has a positive harmonic majorant in $\mathbb{C}_+$. The connection between the Nevanlinna class and the Hardy classes is the following: Each $H^p(\mathbb{C}_+)$, $0 < p \le \infty$, is a subclass of the Nevanlinna class. On the other hand each function of the Nevanlinna class is a quotient of two functions of $H^\infty(\mathbb{C}_+)$. As an application of Corollary 2.2, we have proved the following theorem.

Theorem 2.6 *Let $h \not\equiv 0$ belong to the Nevanlinna class. Assume that $h = g_1 g_2$ where g_1 and g_2 are analytic in $\mathbb{C}_+$ and satisfying the following conditions:*

(i) *There exists a sequence $r_k \to \infty$ such that*

$$\int_0^\pi \log^+ |g_1(re^{i\theta})| \sin\theta d\theta \le \exp\{o(r)\}, \ r = r_k \to \infty. \tag{2.14}$$

(ii) *There exists $H > 0$ such that*

$$\sup_{0<s<H} \int_{-\infty}^{\infty} \frac{\log^+ |g_j(t+is)|}{1+t^2} dt < \infty, \ j = 1, 2. \tag{2.15}$$

Then both g_1 and g_2 belong to the Nevanlinna class.

The following corollary can be derived from Theorem 2.6 by using well-known properties of functions belonging to the Nevanlinna class and the Phragmén-Lindelöf principle.

Corollary 2.7 *Let $h \not\equiv 0$ belong to the Nevanlinna class. Assume that $h = g_1 g_2$ where g_1 and g_2 are analytic in $\mathbb{C}_+$ and g_1 satisfies* (2.14) *of Theorem 2.6. Assume additionally:*

There exists $H > 0$ such that

$$\sup\{|g_1(z)| + |g_2(z)| : \ 0 < \operatorname{Im} z < H\} < \infty.$$

Then the assertion of Theorem C holds.

Evidently (2.14) is less restrictive than (2.13), moreover it relates to only one but not both of functions g_1, g_2. That's why Corollary 2.7 is an amplification of Theorem C.

Condition (2.14) of Theorem 2.6 (of Corollary 2.7 also) cannot be weakened even by replacing $o(r)$ by $O(r)$ as the example $g_1(z) = \exp\{\cos z\}, \ g_2(z) = \exp\{-\cos z\}$ shows. Condition (2.15) cannot also be weakened by replacing it with

$$\exists H > 0, \ \exists \alpha > 2 \qquad \sup_{0<s<H} \int_{-\infty}^{\infty} \frac{\log^+ |g_j(t+is)|}{1+|t|^\alpha} dt < \infty, \ j = 1, 2,$$

as the example $g_1(z) = \exp\{iz^2\}, \ g_2(z) = \exp\{-iz^2\}$ shows. The example $g_1(z) = \exp\{z^2\}, \ g_2(z) = \exp\{-z^2\}$ shows that we cannot relate (2.15) to only one function.

We derived the following factorization theorem from Corollary 2.2 by help of H. Cartan's Second Main Theorem for analytic curves [5]. It shows that, if the number of factors are more than 2, instead of condition (2.14) of Corollary 2.7, we may assume that the factors are connected by a linear equation.

Theorem 2.8 *Let a function $h \not\equiv 0$ belong to $H^\infty(\mathbb{C}_+)$. Suppose that $h = g_1 g_2 \cdots g_n$ where functions $g_j, \ j = 1, 2, \cdots, n, \ n \geq 3$, are analytic in $\mathbb{C}_+$ and satisfy the following conditions:*

(i) *The functions* $g_1, g_2, \cdots, g_{n-1}$ *are linearly independent over* $\mathbb{C}$ *and*

$$g_n = g_1 + g_2 + \cdots + g_{n-1}.$$

(ii) *There exists* $H > 0$ *such that*

$$\sup \left\{ \sum_{j=1}^{n} |g_j(z)| : \ 0 < \operatorname{Im} z < H \right\} < \infty. \tag{2.16}$$

Then there exist real constants $\alpha_j,\ j = 1, 2, \ldots, n$ *such that*

$$g_j(z) e^{i\alpha_j z} \in H^\infty(\mathbb{C}_+),\ j = 1, 2, ..., n.$$

As another application of Corollary 2.2, we obtained a criterion of belonging to the Hardy class up to factor $e^{ikz},\ k \in \mathbb{R}$.

Theorem 2.9 *Let* f *be a function analytic in* $\mathbb{C}_+$. *If*

(i) *the zeros* $\{z_k\}_{k=1}^\infty$ *of* f *satisfy the Blaschke condition, that is,*

$$\sum_{k=1}^{\infty} \frac{\operatorname{Im} z_k}{1 + |z_k|^2} < \infty, \tag{2.17}$$

(ii) *there exists a sequence* $\{r_k\}, r_k \to \infty$, *such that*

$$\int_0^\pi \log^+ |f(re^{i\theta})| \sin\theta d\theta \le \exp\{o(r)\}, \quad r = r_k \to \infty, \tag{2.18}$$

(iii) *there exists* $H > 0$ *such that*

$$\sup_{0<s<H} \int_{-\infty}^{\infty} \frac{\log^- |f(t + is)|}{1 + t^2} dt < \infty, \quad j = 1, 2, \tag{2.19}$$

and

$$\sup_{0<y<H} \|f(\cdot + iy)\|_p < \infty, \tag{2.20}$$

then $f(z)e^{ikz} \in H_p(\mathbb{C}_+)$ *for some* $k \in \mathbb{R}$.

In Chapter 6, we will first prove Theorem 2.6 and Corollary 2.7 and give some examples on the sharpness of assumptions, then we will prove Theorem 2.8 and finally we will prove Theorem 2.9 and we will show that conditions (2.17), (2.18), (2.19), (2.20) are independent, and moreover, (2.18) and (2.19) cannot be substantially weakened.

2.3 Application to generalization of the Titchmarsh convolution theorem

Let M be the set of all finite complex-valued Borel measures $\mu \not\equiv 0$ on $\mathbb{R}$. Set

$$\ell(\mu) = \inf(\operatorname{supp} \mu).$$

The classical Titchmarsh convolution theorem claims that if the measures $\mu_1, \mu_2, \cdots, \mu_n$ belong to M and satisfy

$$\ell(\mu_j) > -\infty,\ j = 1, 2, \cdots, n, \tag{2.21}$$

then

$$\ell(\mu_1 * \mu_2 * \cdots * \mu_n) = \ell(\mu_1) + \ell(\mu_2) + \cdots + \ell(\mu_n), \tag{2.22}$$

where '$*$' denotes the operation of convolution.

Simple examples show that condition (2.21) is essential. One may set

$$\mu_1 = \sum_{m=0}^{\infty} \frac{\delta_{-km}}{m!}, \quad \mu_2 = \sum_{m=0}^{\infty} (-1)^m \frac{\delta_{-km}}{m!}, \quad k > 0, \tag{2.23}$$

where δ_x is the unit measure concentrated at the point x. The Fourier transforms $\hat{\mu}_j$ of the measures μ_j are given by

$$\hat{\mu}_j(z) = \exp\{(-1)^{j+1} e^{-ikz}\}, \quad j = 1, 2.$$

Clearly, $\hat{\mu}_1\hat{\mu}_2 \equiv 1$, and so $\mu_1 * \mu_2 = \delta_0$. We see that $\ell(\mu_1) = \ell(\mu_2) = -\infty$ while $\ell(\mu_1 * \mu_2) = 0$.

It was Y. Domar [6] who first established that condition (2.21) can be replaced by a sufficiently fast decay of μ_j at $-\infty$:

$$\exists \alpha > 2,\ |\mu_j|((-\infty, x)) = O(\exp(-|x|^a)),\ x \to -\infty,\ j = 1, 2, \cdots, n.$$

The best possible condition on decay of μ_j was obtained in [23]:

Theorem D ([23]) *If $\mu_j \in M$ and the condition*

$$|\mu_j|((-\infty, x)) = O(\exp(-c|x| \log |x|)), \quad x \to -\infty, \forall c > 0, \tag{2.24}$$

holds for $j = 1, 2, \cdots, n$, then (2.22) *remains true.*

Observe that measures μ_j in example (2.23) satisfy (2.24) in which '$\forall c > 0$' is replaced with '$c = 1/k$'. Hence, condition (2.24) in Theorem D is sharp.

A simple corollary of Theorem D is that there exist measures $\nu \in M$ with the property that the convolution $\nu^{2*} = \nu * \nu$ is uniquely determined by its values on any fixed half-line $(-\infty, a)$.

Theorem E ([23]) *Suppose $\nu_1, \nu_2 \in M$ and satisfy* (2.24)*. If $\ell(\nu_1) = -\infty$ and $\nu_1^{2*}|_{(-\infty,a)} = \nu_2^{2*}|_{(-\infty,a)}$ for some $a \in \mathbb{R}$, then $\nu_1^{2*} \equiv \nu_2^{2*}$.*

Indeed, set $\mu_1 = \nu_1 + \nu_2$ and $\mu_2 = \nu_1 - \nu_2$. Since ν_1^{2*} and ν_2^{2*} agree on $(-\infty, a)$, we get

$$a \leq \ell(\nu_1^{2*} - \nu_2^{2*}) = \ell((\nu_1 + \nu_2) * (\nu_1 - \nu_2)) = \ell(\mu_1 * \mu_2).$$

Measures μ_1 and μ_2 satisfy (2.24) and at least one of these measures satisfies $\ell(\mu_j) = -\infty$. Hence, one of these measures must be zero, since otherwise Theorem D yields

$$\ell(\mu_1 * \mu_2) = \ell(\mu_1) + \ell(\mu_2) = -\infty.$$

In fact, all n-fold convolutions ν^{n*} have a similar property. Moreover, if $n \geq 3$ then restriction (2.24) can be substantially weakened.

Theorem F ([23]) *Suppose $n \geq 3$, $\nu_1, \nu_2 \in M$ and satisfy the condition*

$$|\nu_j|((-\infty, x)) = O(\exp(-c|x|)), \ x \to -\infty, \ \forall c > 0, j = 1, 2. \tag{2.25}$$

If $\ell(\nu_1) = -\infty$ and $\nu_1^{n*}|_{(-\infty,a)} = \nu_2^{n*}|_{(-\infty,a)}$ for some $a \in \mathbb{R}$, then $\nu_1^{n*} \equiv \nu_2^{n*}$.

Restrictions (2.24) and (2.25) in Theorems E and F are sharp (see, [23]).

Observe that $\nu_1^{n*} - \nu_2^{n*} = (\nu_1 - \nu_2) * (\nu_1 - \epsilon_1\nu_2) * \cdots * (\nu_1 - \epsilon_{n-1}\nu_2)$ where $\epsilon_j = e^{2\pi ij/n}$. Hence, if $n \geq 3$, the difference $\nu_1^{n*} - \nu_2^{n*}$ can be represented as the convolution of linearly dependent measures. One may ask if there is an extension of Theorem D to linearly dependent measures in which restriction (2.24) is weakened.

In this thesis, we extended Theorem D to the measures connected by a linear equation in which restriction (2.24) is replaced by the weaker restriction (2.25). Our result is the following:

Theorem 2.10 *If* $\mu_1, \mu_2, \cdots, \mu_{n-1} \in M,\ n \geq 3,$ *are linearly independent over* $\mathbb{C}$, *satisfy* (2.25) *and*

$$\mu_n = \mu_1 + \mu_2 + \cdots + \mu_{n-1},$$

then

$$\ell(\mu_1 * \mu_2 * \cdots * \mu_n) = \ell(\mu_1) + \ell(\mu_2) + \cdots + \ell(\mu_n), \tag{2.26}$$

In Chapter 7, we will derive Theorem 2.10 from Theorem 2.8 and construct examples which show that (2.25) cannot be weakened by replacing '$\forall$' by '$\exists$'.

Chapter 3

Preliminaries

In this chapter, we recall some definitions and collect some known results which we will need in the sequel.

3.1 Generalized Poisson integral

The function

$$P_q(z,t) = \operatorname{Im}\left\{\frac{1}{\pi}\frac{(1+tz)^q}{(t-z)(1+t^2)^q}\right\}, \quad z \in \mathbb{C}_+,\ t \in \mathbb{R},\ q \in \mathbb{N} \cup \{0\},$$

is called the *generalized Poisson kernel* for the upper half-plane. For $q = 0$ or $q = 1$

$$P_q(z,t) = \operatorname{Im}\left\{\frac{1}{\pi}\frac{1}{(t-z)}\right\} = \frac{1}{\pi}\frac{y}{(x-t)^2+y^2}, \quad z = x+iy \in \mathbb{C}_+,$$

and known as usual *Poisson kernel* for the upper half-plane.

We will apply the following lemma several times:

Lemma 3.1 *The generalized Poisson kernel $P_q(z,t)$ satisfies the estimate*

$$|P_q(z,t)| \le \frac{y}{|t-z|^2}\left(A_q\frac{(1+|z|)^{q-1}}{(1+|t|)^{q-1}} + B_q\frac{(1+|z|)^q}{(1+|t|)^q}\right),\ z = x+iy \in \mathbb{C}_+, \quad (3.1)$$

where A_q and B_q are nonnegative constants.

Proof. For $q = 0,\ 1$ the inequality is trivial (with the choice $A_0 = 0,\ B_0 = 1/\pi,\ A_1 = 1/\pi$ and $B_1 = 0$). Hence, we may assume $q \geq 2$. We have

$$P_q(z,t) = \frac{\operatorname{Im}\{(t-\bar{z})(1+tz)^q\}}{\pi|t-z|^2(1+t^2)^q},$$

$$\begin{aligned}\operatorname{Im}\{(t-\bar{z})(1+tz)^q\} &= \operatorname{Im}\left\{(t-\bar{z})\sum_{k=0}^{q}\binom{q}{k}t^k z^k\right\}\\ &= \sum_{k=0}^{q}\binom{q}{k}t^{k+1}\operatorname{Im}\{z^k\} - \sum_{k=0}^{q}\binom{q}{k}t^k|z|^2\operatorname{Im}\{z^{k-1}\} =: S_1 + S_2.\end{aligned}$$

Using the inequality $|\sin k\theta| \leq k\sin\theta,\ 0 \leq \theta \leq \pi,\ k \in \mathbb{N}$, we obtain

$$|S_1| \leq \sum_{k=1}^{q} k\binom{q}{k}|t|^{k+1}|z|^{k-1}y \leq q(1+|t|)^{q+1}(1+|z|)^{q-1}y, \tag{3.2}$$

$$|S_2| \leq y + \sum_{k=2}^{q}(k-1)\binom{q}{k}|t|^k|z|^k y \leq q(1+|t|)^q(1+|z|)^q y. \tag{3.3}$$

From (3.2), (3.3) and the evident inequality $(1+|t|)^2 \leq 2(1+t^2),\ t \in \mathbb{R}$, we get

$$|P_q(z,t)| \leq \frac{2^q q}{\pi}\cdot\frac{y}{|t-z|^2}\cdot\frac{(1+|z|)^{q-1}\left((1+|t|)+(1+|z|)\right)}{(1+|t|)^q}.$$

□

We also need the following immediate corollary to Lemma 3.1.

Corollary 3.2 *The generalized Poisson kernel $P_q(z,t)$ satisfies the estimate*

$$|P_q(z,t)| \leq C_q\frac{y}{|t-z|^2}\frac{(1+|z|)^q}{(1+|t|)^{q-1}}, \quad z = x+iy \in \mathbb{C}_+, \tag{3.4}$$

where C_q is a positive constant.

We will need the following theorem.

Theorem 3.3 *Let ν be a σ-finite Borel measure on $\mathbb{R}$ and satisfying the following condition:*

There exists $q \in \mathbb{N}$, such that

$$\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1+|t|^{q+1}} < \infty. \tag{3.5}$$

Then the integral

$$u(z) = \int_{-\infty}^{\infty} P_q(z,t)d\nu(t) \tag{3.6}$$

is convergent for any $z \in \mathbb{C}_+$ and represents a harmonic function in $\mathbb{C}_+$.

If $d\nu(t) = f(t)dt$ for some function f continuous on $\mathbb{R}$, then $u(z)$ is continuous on $\overline{\mathbb{C}_+}$ if we define $u(t) = f(t),\ t \in \mathbb{R}$.

The integral (3.6) is called the *generalized Poisson integral of the measure ν*. If the measure ν is absolutely continuous with respect to the Lebesgue measure and $f(t) = d\nu/dt$ then the integral (3.6) can be written in the form

$$u(z) = \int_{-\infty}^{\infty} P_q(z,t)f(t)dt$$

and is called the *generalized Poisson integral of the function f.*

We could not find Theorem 3.3 in the literature therefore we will derive it from the following well-known result:

Theorem G ([16, p.111]) *Let*

$$v(z) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(t)}{(x-t)^2+y^2}$$

where

$$\int_{-\infty}^{\infty} \frac{d|\mu|(t)}{1+t^2} < \infty.$$

If the derivative $\mu'(t_0)$ exists, then $v(t_0+iy) \to \mu'(t_0)$ as $y \to 0$. In particular, if $d\nu(t) = f(t)dt$ for some $f \in L^\infty(\mathbb{R})$, then

$$v(z) \to f(t_0) \quad \text{as } z \to t_0$$

for each continuity point t_0 of f.

Proof of Theorem 3.3. Consider the following functions

$$u_N(z) := \int_{-N}^{N} P_q(z,t)d\nu(t), \quad z \in \mathbb{C}_+,\ N > 0.$$

Since $P_q(z,t)$ is harmonic in $\mathbb{C}_+$ and ν is a σ-finite measure on $\mathbb{R}$, each u_N is harmonic in $\mathbb{C}_+$. Using Corollary 3.2 it follows from (3.5) that u_N converges to u uniformly on compact subsets of $\mathbb{C}_+$ as $N \to \infty$, which implies that u is harmonic in $\mathbb{C}_+$.

Now, let $d\nu(t) = f(t)dt$ for a function f continuous on $\mathbb{R}$. Then condition (3.5) becomes

$$\int_{-\infty}^{\infty} \frac{|f(t)|}{1+|t|^{q+1}}dt < \infty. \tag{3.7}$$

To show that u is continuous on $\overline{\mathbb{C}_+}$, it is enough to show

$$u(z) \to f(t_0) \quad \text{as} \quad z \to t_0,\ \operatorname{Im} z > 0,\ t_0 \in \mathbb{R}.$$

Let $R = 2|t_0| + 1$ and set

$$f_R(t) := f(t)\chi_{[-R,R]}(t), \quad f^R(t) := f(t) - f_R(t).$$

Then we have

$$\begin{aligned} u(z) &= \int_{-\infty}^{\infty} f_R(t)P_q(z,t)dt + \int_{-\infty}^{\infty} f^R(t)P_q(z,t)dt \\ &=: I_1(z) + I_2(z). \end{aligned}$$

First let us show that

$$I_2(z) \to 0 \quad \text{as} \quad z \to t_0.$$

By inequality (3.4), we have

$$|I_2(z)| \le C_q \int_{|t|>R} \frac{y}{|t-z|^2} \frac{(1+|z|)^q}{(1+|t|)^{q-1}} |f(t)|dt.$$

Since for $|z - t_0| < (|t_0| + 1)/2,\ |t| > R$, we have

$$|t-z| \ge |t-t_0| - |t_0 - z| \ge \frac{|t|+1}{2} - \frac{|t|+1}{4} = \frac{|t|+1}{4},$$

and hence

$$|I_2(z)| \leq C_{q,R}\, y \int_{|t|>R} \frac{|f(t)|}{(1+|t|)^{q+1}} dt.$$

Using (3.7), we see that $I_2(z) \to 0$ as $z \to t_0$.

To consider the integral $I_1(z)$, we represent it in the form

$$\begin{aligned} I_1(z) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f_R(t) \operatorname{Im} \left\{ \frac{1}{t-z} - \frac{1}{(1+t^2)^q} \left[\frac{(1+t^2)^q - (1+tz)^q}{t-z} \right] \right\} dt \\ &= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f_R(t)}{(x-t)^2 + y^2} dt - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f_R(t)}{(1+t^2)^q} \operatorname{Im} \left\{ \frac{(1+t^2)^q - (1+tz)^q}{t-z} \right\} dt \\ &=: I_1^1(z) - I_1^2(z). \end{aligned}$$

The integral $I_1^1(z)$ is the usual Poisson integral of the bounded function f_R and converges to $f_R(t_0) = f(t_0)$ as $z \to t_0$ (see, Theorem G, p.17).

To conclude that $u(z)$ is continuous on $\overline{\mathbb{C}_+}$, it only remains to show $I_1^2(z)$ tend to 0 as $z \to t_0$.

Since

$$\begin{aligned} \frac{(1+t^2)^q - (1+tz)^q}{t-z} &= \frac{\sum_{k=1}^{q} \binom{q}{k} t^k (t^k - z^k)}{t-z} \\ &= \sum_{k=1}^{q} \binom{q}{k} t^k \left(\sum_{l=0}^{k-1} t^{(k-1)-l} z^l \right), \end{aligned} \tag{3.8}$$

we have

$$\begin{aligned} \left| \operatorname{Im} \left\{ \frac{(1+t^2)^q - (1+tz)^q}{t-z} \right\} \right| &\leq \sum_{k=2}^{q} \binom{q}{k} |t|^k \left(\sum_{l=1}^{k-1} |t|^{(k-1)-l} l |z|^{l-1} y \right) \\ &\leq C_{q,t_0}\, y. \end{aligned}$$

Therefore

$$|I_1^2(z)| \leq y\, C_{q,t_0} \int_{-R}^{R} \frac{|f(t)|}{(1+t^2)^q} dt \to 0 \text{ as } y \to 0.$$

$\square$

We need the following known result for the representation of a function defined in a strip by a Poisson integral.

Lemma 3.4 *Let v be a function harmonic in a strip $S_h := \{z \in \mathbb{C} : \ 0 < \operatorname{Im} z < h\}$, continuous in its closure $\overline{S_h}$ and satisfying the conditions:*

(i) *There exist two sequences $x_j^+ \to +\infty$ and $x_j^- \to -\infty$ such that*

$$\int_0^h |v(x+iy)| \sin\frac{\pi y}{h} dy = o(e^{\pi|x|/h}), \quad x = x_j^\pm, \ j \to \infty, \tag{3.9}$$

(ii)

$$\int_{-\infty}^{\infty} \{|v(x)| + |v(x+ih)|\} e^{-\pi|x|/h} dx < \infty. \tag{3.10}$$

Then v admits the following representation

$$\begin{aligned} v(z) =& \frac{\sin\frac{\pi y}{h}}{2h} \int_{-\infty}^{\infty} \frac{v(t)dt}{\cosh\frac{\pi(x-t)}{h} - \cos\frac{\pi y}{h}} \\ &+ \frac{\sin\frac{\pi y}{h}}{2h} \int_{-\infty}^{\infty} \frac{v(t+ih)dt}{\cosh\frac{\pi(x-t)}{h} + \cos\frac{\pi y}{h}}, \quad z = x+iy \in S_h. \end{aligned} \tag{3.11}$$

Since we could not find a convenient reference we shall give a proof.

Proof. Denote the expression on the right-hand side of (3.11) by $T(z)$ and let

$$V(z) := v(z) - T(z).$$

It is easy to see that $T(z)$ is a harmonic function in S_h, continuous in $\overline{S_h}$ and takes the same values as $v(z)$ on ∂S_h. Indeed, the change of variables $\zeta = e^{\pi z/h}$, $v(t) = v_1(e^{\pi t/h})$, $v(t+ih) = v_1(-e^{\pi t/h})$, reduces $T(z)$ to the Poisson integral of v_1 for $\mathbb{C}_+$.

By the Symmetry Principle the function $V(z)$ can be extended to the whole complex plane $\mathbb{C}$ as a harmonic function (which we also denote it by $V(z)$) which is odd and $2h$-periodic with respect to $y = \operatorname{Im} z$. This function can be expanded into the absolutely convergent Fourier series

$$V(x+iy) = \sum_{k=1}^{\infty} c_k(x) \sin\frac{k\pi y}{h},$$

where

$$c_k(x) = \frac{1}{h}\int_0^h V(x+iy)\sin\frac{k\pi y}{h}dy,\ k=1,2,\cdots .$$

Since V satisfies the Laplace equation we get

$$c_k''(x) - \left(\frac{k\pi}{h}\right)^2 c_k(x) = 0,\ k=1,2,\cdots .$$

Therefore

$$c_k(x) = c_{1k}e^{k\pi x/h} + c_{2k}e^{-k\pi x/h},\ k=1,2,\cdots \tag{3.12}$$

where c_{1k} and c_{2k} are constants not depending on x.

On the other hand,

$$\begin{aligned} c_k(x) &= \frac{1}{h}\int_0^h v(x+iy)\sin\frac{k\pi y}{h}dy - \frac{1}{h}\int_0^h T(x+iy)\sin\frac{k\pi y}{h}dy \\ &=: c_k^{(1)}(x) - c_k^{(2)}(x). \end{aligned}$$

The elementary inequality $|\sin k\tau| \le k\sin\tau,\ 0\le\tau\le\pi$, implies

$$\begin{aligned} |c_k^{(1)}(x)| &\le \frac{k}{h}\int_0^h |v(x+iy)|\sin\frac{\pi y}{h}dy, \\ |c_k^{(2)}(x)| &\le \frac{k}{h}\int_0^h |T(x+iy)|\sin\frac{\pi y}{h}dy. \end{aligned}$$

Evidently, condition (3.9) implies

$$c_k^{(1)}(x) = o(e^{\pi|x|/h}), \quad x = x_j^{\pm},\ j\to\infty. \tag{3.13}$$

Let us show that

$$c_k^{(2)}(x) = o(e^{\pi|x|/h}), \quad |x|\to\infty. \tag{3.14}$$

Substituting the expression for $T(x+iy)$ and using the Fubini's theorem we obtain

$$\begin{aligned} |c_k^{(2)}(x)| \le& \frac{k}{2h^2}\int_{-\infty}^{\infty}|v(t)|\left\{\int_0^h \frac{\sin^2\frac{\pi y}{h}}{\cosh\frac{\pi(x-t)}{h} - \cos\frac{\pi y}{h}}dy\right\}dt \\ &+ \frac{k}{2h^2}\int_{-\infty}^{\infty}|v(t+ih)|\left\{\int_0^h \frac{\sin^2\frac{\pi y}{h}}{\cosh\frac{\pi(x-t)}{h} + \cos\frac{\pi y}{h}}dy\right\}dt. \end{aligned}$$

A standard calculation shows

$$\int_0^h \frac{\sin^2 \frac{\pi y}{h}}{\cosh \frac{\pi(x-t)}{h} \pm \cos \frac{\pi y}{h}} dy = he^{-\pi|x-t|/h}.$$

Therefore,

$$\begin{aligned}|c_k^{(2)}(x)| &\leq \frac{k}{2h} \int_{-\infty}^{\infty} (|v(t)| + |v(t+ih)|)e^{-\pi|x-t|/h} dt \\ &= \frac{k}{2h} e^{\pi|x|/h} \int_{-\infty}^{\infty} (|v(t)| + |v(t+ih)|)e^{-\pi|t|/h} e^{(\pi/h)(|t|-|x|-|x-t|)} dt.\end{aligned}$$

Since $\exp\{(\pi/h)(|t| - |x| - |x-t|)\}$ is bounded by 1 and tends to 0 as $|x| \to \infty$, we obtain (3.14) from condition (3.10) with help of the Lebesgue Dominated Convergence Theorem.

Equations (3.13), (3.14) and (3.12) show $c_k(x) = 0, \ k = 1, 2, \cdots$. Hence $V(z) = 0$ and $v(z) = T(z)$. □

3.2 Blaschke products

Let $\{\zeta_n\}$ be a finite or infinite sequence from $D := \{\zeta \in \mathbb{C} : \ |\zeta| < 1\}$ satisfying the condition

$$\sum_n (1 - |\zeta_n|) < \infty. \tag{3.15}$$

This condition is called the *Blaschke condition for the unit disk*. Let us form the finite or infinite product

$$\mathfrak{B}(\zeta) = \zeta^m \prod_{\zeta_n \neq 0} \frac{|\zeta_n|}{\zeta_n} \frac{\zeta_n - \zeta}{1 - \overline{\zeta_n}\zeta}, \quad \zeta \in D,$$

where m is the number of ζ_n's equal to 0. A product of such form is called a *Blaschke product* formed by $\{\zeta_n\}$. The following theorem is well-known.

Theorem H ([7, p.19]) *For any sequence $\{\zeta_n\}$ satisfying the Blaschke condition for the unit disk, the Blaschke product formed by $\{\zeta_n\}$ is uniformly convergent on each compact subset of D and hence represents an analytic function in D such that $|\mathfrak{B}(\zeta)| < 1, \ \zeta \in D$. Each ζ_n is a zero of $\mathfrak{B}$, with multiplicity equal to the number of times it occurs in the product, and $\mathfrak{B}$ has no other zeros in D.*

Consider the conformal transformation

$$\zeta(z) := \frac{z-i}{z+i}$$

which maps $\mathbb{C}_+$ onto D. For each sequence $\{z_n\} \subset \mathbb{C}_+$ we obtain a sequence $\{\zeta_n\} \subset D$ such that $\zeta_n = \zeta(z_n)$ and condition (3.15) is equivalent to

$$\sum_n \frac{\operatorname{Im} z_n}{1+|z_n|^2} < \infty.$$

This condition is called the *Blaschke condition for the upper half-plane*. Thus,

$$B(z) := \mathfrak{B}(\zeta(z)) = \left(\frac{z-i}{z+i}\right)^m \prod_{z_n \neq i} \frac{|z_n^2+1|}{z_n^2+1} \frac{z-z_n}{z-\overline{z_n}}$$

is uniformly convergent on each compact subset of $\mathbb{C}_+$ and hence represents an analytic function in $\mathbb{C}_+$ such that $|B(z)| < 1,\ z \in \mathbb{C}_+$. Each z_n is a zero of B, with multiplicity equal to the number of times it occurs in the product, and B has no other zeros in $\mathbb{C}_+$.

3.3 Hardy classes and the Nevanlinna class

The *Hardy class* $H^p(\mathbb{C}_+),\ 0 < p < \infty$, is the class of all functions f analytic in $\mathbb{C}_+$ and satisfying the condition

$$\sup_{0<y<\infty} \int_{-\infty}^{\infty} |f(x+iy)|^p dx < \infty.$$

By $H^\infty(\mathbb{C}_+)$ is denoted the set of all bounded analytic functions in $\mathbb{C}_+$.

The following factorization theorem is a standard tool in the theory of H^p classes.

Theorem I ([7, p.191]) *Let $f \not\equiv 0$ be a function belonging to $H^p(\mathbb{C}_+),\ 0 < p \leq \infty$. Then the zeros of f satisfy the Blaschke condition and f admits the following factorization*

$$f(z) = B(z)g(z), \quad z \in \mathbb{C}_+,$$

where B is the Blaschke product for the upper half-plane formed by the zeros of f in $\mathbb{C}_+$ and g is a non-vanishing function of $H^p(\mathbb{C}_+)$.

A function f analytic in the upper half-plane is said to belong to the *Nevanlinna class* if $\log|f|$ has a positive harmonic majorant in $\mathbb{C}_+$. It is known that each $H^p(\mathbb{C}_+),\ 0 < p \leq \infty$, is contained in the Nevanlinna class (see, [7, p.16]).

This class of functions is closely related to the $H^\infty(\mathbb{C}_+)$ as the following theorem shows.

Theorem J ([7, p.16]) *Let f be a function analytic in $\mathbb{C}_+$. Then f belong to the Nevanlinna class if and only if f can be written in the form F_1/F_2, where $F_j,\ j = 1, 2$, belong to $H^\infty(\mathbb{C}_+)$, $|F_j(z)| < 1,\ z \in \mathbb{C}_+,\ j = 1, 2$ and F_2 does not vanish in $\mathbb{C}_+$.*

This theorem and the previous one allow us to conclude the following.

Corollary 3.5 *Let $f \not\equiv 0$ be a function belonging to the Nevanlinna class. Then the zeros of f satisfy the Blaschke condition and f admits the following factorization*

$$f(z) = B(z)F(z), \quad z \in \mathbb{C}_+,$$

where B is the Blaschke product formed by the zeros of f in $\mathbb{C}_+$ and F is a non-vanishing function of the Nevanlinna class.

The following theorem gives a complete description of the Nevanlinna class.

Theorem K ([16, p.119]) *The Nevanlinna class consists of functions representable in the form*

$$f(z) = B(z)e^{i(k_1 z + k_2)} \exp\left\{\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1 + tz}{(t - z)(1 + t^2)} d\nu(t)\right\},$$

where B is a Blaschke product, k_1 and k_2 are real constants and ν is a real-valued Borel measure satisfying the condition

$$\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1 + t^2} < \infty.$$

We will use the following result of [23] several times.

Theorem L ([23]) *If a function $Q \not\equiv 0$ belongs to $H^\infty(\mathbb{C}_+)$, then for any $K > 0$*

$$\sup_{0<s<K} \int_{-\infty}^{\infty} \frac{\log^+ |1/Q(t+is)|}{1+t^2} dt < \infty.$$

Using Theorem J, we derive the following corollary from Theorem L.

Corollary 3.6 *If a function $Q \not\equiv 0$ belongs to the Nevanlinna class, then for any $K > 0$*

$$\sup_{0<s<K} \int_{-\infty}^{\infty} \frac{|\log |Q(t+is)||}{1+t^2} dt < \infty.$$

3.4 Carleman's and Nevanlinna's formulas

The following integral formula is called the *Carleman's formula.* It connects the modulus and the zeros of a function analytic in $\mathbb{C}_+$. This formula has important applications in the theory of entire functions.

Theorem M ([18, p.224]) *Let F be a function analytic in the region $\{z \in \mathbb{C} : 0 < \rho \le |z| \le R,\ \operatorname{Im} z \ge 0\}$ and $a_k = r_k e^{i\theta_k}$ be its zeros. Then*

$$\sum_{\rho<r_k<R} \left(\frac{1}{r_k} - \frac{r_k}{R^2} \right) \sin\theta_k = \frac{1}{\pi R} \int_0^\pi \log |F(Re^{i\theta})| \sin\theta d\theta$$
$$+ \frac{1}{2\pi} \int_\rho^R \left(\frac{1}{x^2} - \frac{1}{R^2} \right) \log |F(x)F(-x)| dx + A_\rho(F,R),$$

where

$$A_\rho(F,R) = -\operatorname{Im} \left\{ \frac{1}{2\pi} \int_0^\pi \log F(\rho e^{i\theta}) \left(\frac{\rho e^{i\theta}}{R^2} - \frac{e^{-i\theta}}{\rho} \right) d\theta \right\}.$$

Remark: If F is analytic for $|z| \ge \rho,\ \operatorname{Im} z \ge 0$, the quantity $A_\rho(F,R)$ is bounded for $R > \rho$ and as $R \to \infty$, we have the limit

$$A_\rho(F,\infty) = \operatorname{Im} \left\{ \frac{1}{2\pi} \int_0^\pi \log F(\rho e^{i\theta}) \frac{e^{-i\theta}}{\rho} d\theta \right\}.$$

We will also use the following formula for a harmonic function in a half-disk which is called the *Nevanlinna's formula.*

Theorem N ([19, p.193]) *Let u be a function harmonic in the half-disk $D_R^+ := \{z \in \mathbb{C} : |z| < R, \operatorname{Im} z > 0\}$ and continuous in its closure. Then*

$$u(z) = \frac{1}{2\pi}\int_0^\pi \frac{(R^2-r^2)4Rr\sin\theta\sin\varphi}{|Re^{i\theta}-z|^2|Re^{i\theta}-\bar{z}|^2}u(Re^{i\theta})d\theta$$
$$+\frac{r\sin\varphi}{\pi}\int_{-R}^{R}\left(\frac{1}{|t-z|^2}-\frac{R^2}{|R^2-tz|^2}\right)u(t)dt, \quad z = re^{i\varphi} \in D_R^+.$$

3.5 Compactness Principle for harmonic functions

Recall that a family of analytic or harmonic functions in a region Ω is said to be *normal* if every sequence contains a subsequence that converges uniformly on every compact set $E \subset \Omega$.

We have the following well-known Compactness Principle for analytic functions (see, e.g., [1, Ch.5]).

Theorem O *A family F of analytic functions in a region Ω is normal if and only if the functions in F are uniformly bounded on every compact subset E of Ω.*

The following analogue theorem for harmonic functions is an immediate corollary of the previous one.

Theorem P *Let Ω be a simply connected region. A family F of harmonic functions in Ω is normal if and only if the functions in F are uniformly bounded on every compact subset E of Ω.*

3.6 Nevanlinna characteristics

Let $f(z)$ be a function meromorphic in the disk D_R, that is, let the only singularities of f be poles.

Denote by $n_f(r,\infty)$ the number of poles of f, taking into account their multiplicities, in the closed disk $\overline{D_r}$ for $0 \le r < R$.

Following R. Nevanlinna, let us introduce

$$N(r,f) := \int_0^r \frac{n_f(t,\infty) - n_f(0,\infty)}{t} dt + n(0,\infty)\log r, \quad 0 \le r < R.$$
$$m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \quad 0 \le r < R.$$

The function

$$T(r,f) := m(r,f) + N(r,f), \quad 0 \le r < R.$$

is called the *Nevanlinna Characteristic*.

The following theorem which is known as *Jensen formula*, connects the distribution of zeros and poles of a meromorphic function with its growth.

Theorem Q ([19, p.12]) *Let f be a meromorphic function in the disk D_R. Then*

$$T(r,f) = T\left(r, \frac{1}{f}\right) + C, \quad 0 \le r < R.$$

Here C is a constant.

The followings are the main properties of the Nevanlinna characteristics and they are easy to derive (see, e.g. [13, Ch.1, §6], [19, Ch.2], [22, Ch.6, §2.5]).

Let f be a meromorphic function in the disk D_R. Then for $r < R$

- $T\left(r, \dfrac{af+b}{cf+d}\right) = T(r,f) + O(1), \quad r \to R,\ ad - bc \ne 0.$
- $T\left(r, \sum_{k=1}^{n} f_k\right) \le \sum_{k=1}^{n} T(r,f_k) + O(1), \quad r \to R.$
- $T\left(r, \prod_{k=1}^{n} f_k\right) \le \sum_{k=1}^{n} T(r,f_k) + O(1), \quad r \to R.$
- $T\big(r, f^n\big) = nT(r,f).$

Let f be an analytic function in D_R and $M(r,f) := \max\{|f(z)| : \ |z| \leq r\}$. Then for $0 \leq r < \rho < R$

- $\log^+ M(r,f) \leq \dfrac{\rho+r}{\rho-r} m(\rho,f) = \dfrac{\rho+r}{\rho-r} T(\rho,f)$.

We will apply the following immediate corollary of H. Cartan's Second Main Theorem for analytic curves to prove Theorem 2.8.

Theorem R (**[5]**) *Let $f_1, f_2, \cdots, f_n$, $n \geq 3$, be functions analytic in the unit disc whose zeros satisfy the Blaschke condition. If $f_1, f_2, \cdots, f_{n-1}$, are linearly independent over $\mathbb{C}$ and*

$$f_n = f_1 + f_2 + \cdots + f_{n-1},$$

then

$$T\left(r, \frac{f_j}{f_n}\right) = O\left(\log \frac{1}{1-r}\right), \quad r \to 1, \ j = 1, \cdots, n-1.$$

3.7 Titchmarsh convolution theorem

Let M be the set of all finite complex-valued Borel measures μ on $\mathbb{R}$ such that $|\mu|(\mathbb{R}) \not\equiv 0$. For each measure $\mu \in M$, the *support* $\operatorname{supp}\mu$ of μ is defined as the complement of the largest open set O such that $\mu(O) = 0$.

Set

$$\ell(\mu) := \inf(\operatorname{supp}\mu).$$

Note that ℓ may be finite or infinite, but since M consists of measures not identically zero, ℓ cannot be $+\infty$.

The *convolution* $\mu * \nu$ of the measures μ and ν is defined as

$$(\mu * \nu)(E) := \int_{-\infty}^{\infty} \mu(E-t) d\nu(t), \quad \text{for each Borel set } E \subset \mathbb{R},$$

and the *Fourier transform* $\hat{\mu}(z)$ of μ is defined as

$$\hat{\mu}(z) = \int_{-\infty}^{\infty} e^{itz} d\mu(t), \quad z \in \mathbb{R}.$$

The following well-known property relating the convolution and Fourier transform is important

$$(\widehat{\mu * \nu})(z) = \hat{\mu}(z)\hat{\nu}(z), \quad z \in \mathbb{R}.$$

It is evident from the definitions that the inequality

$$\ell(\mu * \nu) \geq \ell(\mu) + \ell(\nu) \tag{3.16}$$

holds without any restrictions on $\mu,\ \nu \in M$. The classical Titchmarsh convolution theorem says

Theorem S ([16, Ch.VI F],[19, §16.2]) *If* $\mu,\ \nu \in M$ *and* $\ell(\mu) > -\infty$, $\ell(\nu) > -\infty$ *then*

$$\ell(\mu * \nu) = \ell(\mu) + \ell(\nu).$$

Remark. This theorem had been proved by Titchmarsh [24] for L_1-functions instead of measures from M, but its proof extends to measures without any difficulty.

We will need the following well-known corollary of the Paley-Wiener theorem

Theorem T ([20, p.206]) *Let* μ *be a finite Borel measure whose Fourier transform can be analytically continued to the whole plane* $\mathbb{C}$ *and the extended function* $\hat{\mu}$ *satisfies the condition*

$$|\hat{\mu}(z)| \leq Ae^{B|z|}, \quad z \in \mathbb{C},$$

for some positive constants A *and* B. *Then* $\operatorname{supp}\mu \subset [-B, B]$.

Chapter 4

Auxiliary results

4.1 Estimates for means of Blaschke products and Poisson integrals

Lemma 4.1 *Let $B(z)$ be a Blaschke product. Then*

$$\int_0^\pi \log^+ \frac{1}{|B(re^{i\theta})|} \sin\theta d\theta = O(r), \quad r \to \infty.$$

Proof. Let $\{z_n\}$ be the zeros of $B(z)$. Without loss of generality we can assume $i \notin \{z_n\}$. We write B in the form $B = B_1 B_2$ where

$$B_1(z) = A \prod_{|z_n|<1} \frac{|z_n^2+1|}{z_n^2+1} \frac{z-z_n}{z-\bar{z}_n},$$
$$B_2(z) = \frac{1}{A} \prod_{|z_n|\geq 1} \frac{|z_n^2+1|}{z_n^2+1} \frac{z-z_n}{z-\bar{z}_n},$$

and A is chosen to make $B_2(0) = 1$.

First consider $B_1(z)$. Evidently, $B_1(z)$ is analytic in $\{z \in \mathbb{C} : \; |z| > 1\}$ and $\lim_{z\to\infty} |B_1(z)| = A$.

Hence, we get

$$\int_0^{\pi} \log^+ \frac{1}{|B_1(re^{i\theta})|} \sin\theta d\theta = O(1), \quad r \to \infty. \tag{4.1}$$

Now put for $0 < h \leq 1/2$

$$B_2^{(h)}(z) := B_2(z + ih).$$

Evidently, $B_2^{(h)}$ is analytic in $\overline{\mathbb{C}_+}$. Applying Carleman's formula (See, Theorem M, p.25) to $B_2^{(h)}$ in the region $\{z \in \mathbb{C} : \ h \leq |z| \leq r, \ \operatorname{Im} z \geq 0\}$, we have

$$\sum_{h<|a_{k,h}|<r} \left(\frac{1}{|a_{k,h}|} - \frac{|a_{k,h}|}{r^2} \right) \sin\theta_{k,h} = \frac{1}{\pi r} \int_0^{\pi} \log|B_2^{(h)}(re^{i\theta})| \sin\theta d\theta$$
$$+ \frac{1}{2\pi} \int_h^r \left(\frac{1}{t^2} - \frac{1}{r^2} \right) \log|B_2^{(h)}(t) B_2^{(h)}(-t)| dt + A_h(B_2^{(h)}, r), \tag{4.2}$$

where $a_{k,h} = |a_{k,h}| e^{i\theta_{k,h}}$ are the zeros of $B_2^{(h)}(z)$ in $\mathbb{C}_+$ and

$$\begin{aligned} A_h(B_2^{(h)}, r) &= -\operatorname{Im} \left\{ \frac{1}{2\pi} \int_0^{\pi} \log B_2^{(h)}(he^{i\theta}) \left(\frac{he^{i\theta}}{r^2} - \frac{e^{-i\theta}}{h} \right) d\theta \right\} \\ &= -\operatorname{Im} \left\{ \frac{1}{2\pi} \int_0^{\pi} \log B_2(he^{i\theta} + ih) \left(\frac{he^{i\theta}}{r^2} - \frac{e^{-i\theta}}{h} \right) d\theta \right\}. \end{aligned} \tag{4.3}$$

Since the term in the left hand side of (4.2) is nonnegative and the second term in the right hand side is non-positive, we have

$$\begin{aligned} \frac{1}{\pi r} \int_0^{\pi} \log^+ \frac{1}{|B_2^{(h)}(re^{i\theta})|} \sin\theta d\theta &= \frac{1}{\pi r} \int_0^{\pi} \log^+ \frac{1}{|B_2(re^{i\theta} + ih)|} \sin\theta d\theta \\ &\leq A_h(B_2^{(h)}, r). \end{aligned} \tag{4.4}$$

Since B_2 is analytic in $|z| \leq 1/2$ and $B_2(0) = 1$, we have the power series expansion:

$$B_2(z) = 1 + cz + O(|z|^2) \quad \text{for } |z| \leq 1/2, \ z \to 0.$$

Hence, for h is small enough, we have

$$B_2(he^{i\theta} + ih) = 1 + che^{i\theta} + cih + O(h^2), \quad h \to 0.$$

Using the power series expansion for $\log(1+z) = z - \frac{z^2}{2} + \cdots$, we obtain

$$\log B_2(he^{i\theta} + ih) = che^{i\theta} + cih + O(h^2), \quad h \to 0. \tag{4.5}$$

If we substitute (4.5) into (4.3), we get

$$\begin{aligned} A_h(B_2^{(h)}, r) &= -\operatorname{Im}\left\{\frac{1}{2\pi}\int_0^\pi \{-c - ice^{-i\theta} + O(h)\}d\theta\right\} \\ &= -\frac{\operatorname{Im} c}{2} + O(h), \quad h \to 0, \end{aligned}$$

which implies

$$\lim_{h\to 0} A_h(B_2^{(h)}, r) = -\frac{\operatorname{Im} c}{2} = -\frac{\operatorname{Im}\{B_2'(0)\}}{2} \leq \frac{|B_2'(0)|}{2}. \tag{4.6}$$

Now, applying Fatou's lemma, we obtain

$$\frac{1}{\pi r}\int_0^\pi \log^+ \frac{1}{|B_2(re^{i\theta})|}\sin\theta d\theta \leq \liminf_{h\to 0} \frac{1}{\pi r}\int_0^\pi \log^+ \frac{1}{|B_2(re^{i\theta} + ih)|}\sin\theta d\theta. \tag{4.7}$$

Putting (4.4), (4.6) and (4.7) together, we get

$$\frac{1}{\pi r}\int_0^\pi \log^+ \frac{1}{|B_2(re^{i\theta})|}\sin\theta d\theta \leq \frac{|B_2'(0)|}{2},$$

and, hence

$$\int_0^\pi \log^+ \frac{1}{|B_2(re^{i\theta})|}\sin\theta d\theta = O(r), \quad r \to \infty. \tag{4.8}$$

So,

$$\begin{aligned} \int_0^\pi \log^+ \frac{1}{|B(re^{i\theta})|}\sin\theta d\theta &= \int_0^\pi \log^+ \frac{1}{|B_1(re^{i\theta})B_2(re^{i\theta})|}\sin\theta d\theta \\ &\leq \int_0^\pi \log^+ \frac{1}{|B_1(re^{i\theta})|}\sin\theta d\theta + \int_0^\pi \log^+ \frac{1}{|B_2(re^{i\theta})|}\sin\theta d\theta. \end{aligned}$$

Using (4.1), and (4.8), we get,

$$\int_0^\pi \log^+ \frac{1}{|B(re^{i\theta})|}\sin\theta d\theta = O(r), \quad r \to \infty.$$

□

Lemma 4.2 *Let v be a function of the form*

$$v(z) = \int_{-\infty}^{\infty} P_q(z,t)d\mu(t), \quad z = x+iy \in \mathbb{C}_+, \quad q \in \mathbb{N} \cup \{0\},$$

where μ is a real-valued Borel measure satisfying

$$\int_{-\infty}^{\infty} \frac{d|\mu|(t)}{1+|t|^{q+1}} < \infty. \tag{4.9}$$

Then

$$\int_0^{\pi} |v(re^{i\theta})| \sin\theta d\theta = O(r^q), \quad r \to \infty. \tag{4.10}$$

Proof. We have by the definition of v that

$$\int_0^{\pi} |v(re^{i\theta})| \sin\theta d\theta \le \int_0^{\pi} \sin\theta \left\{ \int_{-\infty}^{\infty} |P_q(re^{i\theta},t)|\, d|\mu|(t) \right\} d\theta.$$

Without loss of generality we can assume $r > 1$. By Fubini's theorem and estimate (3.1) we have

$$\int_0^{\pi} |v(re^{i\theta})| \sin\theta d\theta =$$
$$r^q \int_{-\infty}^{\infty} \left(\frac{\widetilde{A}_q}{(1+|t|)^{q-1}} + \frac{\widetilde{B}_q r}{(1+|t|)^q} \right) \left\{ \int_0^{\pi} \frac{\sin^2\theta}{r^2+t^2-2tr\cos\theta} d\theta \right\} d|\mu|(t). \tag{4.11}$$

A standard calculation shows that

$$\int_0^{\pi} \frac{\sin^2\theta d\theta}{r^2+t^2-2rt\cos\theta} = \frac{\pi}{2} \min\left(\frac{1}{r^2}, \frac{1}{t^2} \right). \tag{4.12}$$

Substituting this into (4.11), we get

$$\begin{aligned} \int_0^{\pi} |v(re^{i\theta})| \sin\theta d\theta \le & r^q \int_{|t|\le r} \left(\frac{\widetilde{A}_q}{(1+|t|)^{q-1} r^2} + \frac{\widetilde{B}_q}{(1+|t|)^q r} \right) d|\mu|(t) \\ & + r^q \int_{|t|>r} \left(\frac{\widetilde{A}_q}{(1+|t|)^{q-1} t^2} + \frac{\widetilde{B}_q r}{(1+|t|)^q t^2} \right) d|\mu|(t) \\ \le & D_q r^q \int_{-\infty}^{\infty} \frac{d|\mu|(t)}{1+|t|^{q+1}}, \end{aligned}$$

with the aid of (4.9) this implies (4.10). $\square$

4.2 A representation theorem

Lemma 4.3 *Let u be a function harmonic in $\mathbb{C}_+$ and satisfying the following condition:*

There exist $H > 0$ and $\alpha > 0$ such that

$$\sup_{0<s<H} \int_{-\infty}^{\infty} \frac{|u(t+is)|}{1+|t|^\alpha} dt < \infty. \tag{4.13}$$

Then u admits the representation

$$u(z) = \int_{-\infty}^{\infty} P_q(z,t) d\nu(t) + U(z), \tag{4.14}$$

where q and ν are as in Theorem 2.1 and U is a function harmonic in $\mathbb{C}$ such that $U(x) = 0,\ x \in \mathbb{R}$.

Proof. Consider the following family of Borel measures on $\mathbb{R}$:

$$\sigma_s(E) = \int_E \frac{u(t+is)}{1+|t|^\alpha} dt, \quad E \subset \mathbb{R},\ 0 < s < H.$$

By (4.13), each sequence $\{\sigma_{s_k}\},\ \lim_{k\to\infty} s_k = 0$, contains a subsequence (which we also denote by $\{\sigma_{s_k}\}$) weak-star convergent to a finite Borel measure σ on $\overline{\mathbb{R}}$ (2-point compactification of $\mathbb{R}$). Hence, noting that

$$\lim_{t\to\pm\infty} (1+|t|^\alpha) P_q(z,t) = \begin{cases} 0 & \text{if} \quad \alpha < q+1 \\ \dfrac{(\pm 1)^{q+1}}{\pi} \operatorname{Im}\{z^q\} & \text{if} \quad \alpha = q+1 \end{cases},$$

we get

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} u(t+is_k) P_q(z,t) dt = \int_{-\infty}^{\infty} (1+|t|^\alpha) P_q(z,t) d\sigma(t), \quad \text{if} \quad \alpha < q+1, \tag{4.15}$$

and

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} u(t+is_k) P_q(z,t) dt = \int_{-\infty}^{\infty} (1+|t|^\alpha) P_q(z,t) d\sigma(t) + \frac{1}{\pi} \operatorname{Im}\{z^q\} \left[\sigma(\{\infty\}) + (-1)^{q+1} \sigma(\{-\infty\})\right], \quad \text{if} \quad \alpha = q+1. \tag{4.16}$$

Joining (4.15) and (4.16) and setting $d\nu(t) = (1 + |t|^\alpha)d\sigma(t)$, we obtain

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} u(t + is_k)P_q(z,t)dt = \int_{-\infty}^{\infty} P_q(z,t)d\nu(t) + A \ \mathrm{Im}\{z^q\}, \tag{4.17}$$

for some constant A.

Consider the following family of functions:

$$U_s(z) = u(z+is) - \int_{-\infty}^{\infty} u(t+is)P_q(z,t)dt, \ z = x+iy \in \mathbb{C}_+, \ 0 < s < \frac{H}{2}. \tag{4.18}$$

From (4.13) and Theorem 3.3 it follows that for each s with $0 < s < H/2$ the function U_s is harmonic in $\mathbb{C}_+$ and becomes continuous in $\overline{\mathbb{C}_+}$ if we set $U_s(x) = 0$ for $x \in \mathbb{R}$. By the Symmetry Principle, it can be extended harmonically to $\mathbb{C}$ and then fulfils $U_s(z) = -U_s(\bar{z})$ for $z \in \mathbb{C}$.

Now let us show that, for any fixed R, the family $\{U_s : \ 0 < s < H/2\}$ is uniformly bounded in the rectangle

$$\Pi_R = \left\{ z \in \mathbb{C} : \ |\operatorname{Re} z| \le R, \ |\operatorname{Im} z| \le \frac{H}{4} \right\}.$$

To this end, we shall first show that

$$\int_{-\infty}^{\infty} \frac{|U_s(x+iy)|}{1+|x|^{q+1}}dx \le C \quad \text{ for } |y| \le \frac{H}{2}, \tag{4.19}$$

for some constant[1] C which does not depend on y and s.

Note that it is enough to show (4.19) for $0 < y \le H/2$. Using Fubini's theorem and (4.18) we get

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{|U_s(x+iy)|}{1+|x|^{q+1}}dx &\le \int_{-\infty}^{\infty} \frac{|u(x+i(y+s))|}{1+|x|^{q+1}}dx \\ &+ \int_{-\infty}^{\infty} |u(t+is)| \left\{ \int_{-\infty}^{\infty} \frac{|P_q(z,t)|}{1+|x|^{q+1}}dx \right\} dt =: I_1 + I_2. \end{aligned} \tag{4.20}$$

By (4.13), I_1 is bounded by a constant not depending on y and s. By (3.1) we

[1] Here and in what follows letters A, B, C with or without subscripts denote the various positive constants.

obtain for $0 < y \leq H/2$

$$\begin{aligned}
I_2 \leq & A_q \int_{-\infty}^{\infty} \frac{|u(t+is)|}{(1+|t|)^{q-1}} \left\{ \int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} \frac{(1+|z|)^{q-1}}{(1+|x|)^{q+1}} dx \right\} dt \\
& + B_q \int_{-\infty}^{\infty} \frac{|u(t+is)|}{(1+|t|)^{q}} \left\{ \int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} \frac{(1+|z|)^{q}}{(1+|x|)^{q+1}} dx \right\} dt \\
\leq & A_{q,H} \int_{-\infty}^{\infty} \frac{|u(t+is)|}{(1+|t|)^{q-1}} \left\{ \int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} \frac{1}{1+x^2} dx \right\} dt \\
& + B_{q,H} \int_{-\infty}^{\infty} \frac{|u(t+is)|}{(1+|t|)^{q}} \left\{ \int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} \frac{1}{1+|x|} dx \right\} dt. \qquad (4.21)
\end{aligned}$$

Since

$$\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} \frac{1}{1+x^2} dx = \frac{\pi(y+1)}{t^2+(y+1)^2} \leq \frac{\pi(H+2)}{2(t^2+1)}, \qquad (4.22)$$

the Schwarz inequality gives

$$\begin{aligned}
& \int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} \frac{1}{1+|x|} dx \\
& \quad \leq \left(\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} \frac{1}{1+x^2} dx \right)^{1/2} \left(\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} dx \right)^{1/2} \\
& \quad \leq \frac{\pi\sqrt{H+2}}{1+|t|}. \qquad (4.23)
\end{aligned}$$

Inserting (4.22) and (4.23) into (4.21) and using (4.13), we conclude that I_2 is bounded by a constant not depending on y and s. This proves (4.19).

Since $|U_s|$ is subharmonic function, we have for any $\rho > 0$

$$\begin{aligned}
|U_s(z)| & \leq \frac{1}{\pi\rho^2} \int\int_{|\xi+i\eta-z|\leq\rho} |U_s(\xi+i\eta)| d\xi d\eta \\
& = \frac{1}{\pi\rho^2} \int\int_{|\xi+i\eta-z|\leq\rho} (1+|\xi|^{q+1}) \frac{|U_s(\xi+i\eta)|}{1+|\xi|^{q+1}} d\xi d\eta \\
& \leq \frac{1+(|z|+\rho)^{q+1}}{\pi\rho^2} \int_{\operatorname{Im} z-\rho}^{\operatorname{Im} z+\rho} \left\{ \int_{\operatorname{Re} z-\rho}^{\operatorname{Re} z+\rho} \frac{|U_s(\xi+i\eta)|}{1+|\xi|^{q+1}} d\xi \right\} d\eta. \qquad (4.24)
\end{aligned}$$

With the choice $\rho = H/4$ and by the aid of (4.19) we obtain for $z \in \Pi_R$

$$\begin{aligned}
|U_s(z)| & \leq \frac{1+(R+H/2)^{q+1}}{\pi(H/4)^2} \int_{-H/2}^{H/2} \left\{ \int_{-\infty}^{\infty} \frac{|U_s(\xi+i\eta)|}{1+|\xi|^{q+1}} d\xi \right\} d\eta \\
& \leq \frac{1+(R+H/2)^{q+1}}{\pi(H/4)^2} H\, C.
\end{aligned}$$

Thus the family $\{U_s : \ 0 < s < H/2\}$ is uniformly bounded in Π_R.

Let $\{s_k\}_{k=1}^{\infty}$ be a sequence such that (4.17) holds. By the well-known Compactness Principle for harmonic functions (see, Theorem P, p.26), we can extract a subsequence (which we also denote by $\{s_k\}$) such that the sequence $\{U_{s_k}\}_{k=1}^{\infty}$ is uniformly convergent on any compact subset of the strip $\{z \in \mathbb{C} : \ |\operatorname{Im} z| < H/4\}$. Let U be the limiting function. Evidently U is harmonic in the strip $\{z \in \mathbb{C} : \ |\operatorname{Im} z| < H/4\}$ and $U(x) = 0, \ x \in \mathbb{R}$. With the choice $s = s_k$ in (4.18) and taking limit as $k \to \infty$ we obtain

$$U(z) = u(z) - \int_{-\infty}^{\infty} P_q(z,t)d\nu(t) - A \ \operatorname{Im}\{z^q\}, \tag{4.25}$$

for $0 < \operatorname{Im} z < H/4$. The right hand side of (4.25) is a harmonic function in $\mathbb{C}_+$ therefore U can be harmonically extended to $\mathbb{C}_+$. Since $U(x) = 0, \ x \in \mathbb{R}$, it can be harmonically extended to $\mathbb{C}$. To derive formula (4.14), we only have to replace $U(z) + A \ \operatorname{Im}\{z^q\}$ by $U(z)$. □

4.3 A criterion of belonging to $H^{\infty}(\mathbb{C}_+)$ up to an exponential factor for functions of the Nevanlinna class

Lemma 4.4 *Let $f \not\equiv 0$ belong to the Nevanlinna class and satisfy the following condition:*

There exists $H > 0$ such that

$$\sup\{|f(z)| : \ 0 < \operatorname{Im} z < H\} < \infty. \tag{4.26}$$

Then there exists a real constant α such that $f(z)e^{i\alpha z} \in H^{\infty}(\mathbb{C}_+)$.

Proof. Since f belongs to the Nevanlinna class, f can be written in the form (see, Theorem K, p.24)

$$f(z) = B(z)e^{i(az+b)} \exp\left\{\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1+tz}{(t-z)(1+t^2)}d\nu(t)\right\}, \tag{4.27}$$

where B is a Blaschke product, a and b are real constants and ν is a real-valued Borel measure satisfying the condition

$$\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1+t^2} < \infty.$$

We claim that $f(z)e^{-i(a-\epsilon)z} \in H^\infty(\mathbb{C}_+)$ for any fixed $\epsilon > 0$. Evidently, this function is bounded in $\{z \in \mathbb{C} : \ 0 < \operatorname{Im} z < H\}$ by condition (4.26). By (4.27) for $y \geq H$ and any fixed $N > 1$ we have

$$\begin{aligned}
|f(z)e^{-iaz}| &\leq \exp\left\{\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{d\nu^+(t)}{(x-t)^2+y^2}\right\} \\
&\leq \exp\left\{\frac{1}{\pi y}\int_{-N}^{N} d\nu^+(t) + \frac{y}{\pi}\int_{|t|\geq N}\frac{2t^2}{1+t^2}\frac{d\nu^+(t)}{(x-t)^2+y^2}\right\} \\
&\leq \exp\left\{\frac{1}{\pi H}\int_{-N}^{N} d\nu^+(t) + \frac{y}{\pi}\max_{|t|\geq 1}\frac{2t^2}{(x-t)^2+y^2}\int_{|t|\geq N}\frac{d\nu^+(t)}{1+t^2}\right\} \\
&= \exp\left\{\frac{1}{\pi H}\int_{-N}^{N} d\nu^+(t) + \frac{2(x^2+y^2)}{\pi y}\int_{|t|\geq N}\frac{d\nu^+}{1+t^2}\right\}, \quad z = x+iy.
\end{aligned}$$

Since N can be taken arbitrarily large, we get

$$|f(z)e^{-iaz}| = e^{o(|z|^2)}, \quad |z| \to \infty, \ \operatorname{Im} z \geq H,$$

and

$$|f(z)e^{-iaz}| = e^{o(|z|)}, \quad |z| \to \infty, \ |\pi/2 - \arg z| \leq \pi/4. \tag{4.28}$$

Evidently, $|f(z)e^{-i(a-\epsilon)z}| = e^{o(|z|^2)}, \quad |z| \to \infty, \ \operatorname{Im} z \geq H$. It follows from (4.28) and (4.26) that $f(z)e^{-i(a-\epsilon)z}$ is bounded on the boundary of the regions $\{z \in \mathbb{C} : \ \operatorname{Re} z > 0, \ \operatorname{Im} z > H\}$ and $\{z \in \mathbb{C} : \ \operatorname{Re} z < 0, \ \operatorname{Im} z > H\}$. Applying the Phragmén-Lindelöf principle (see, e.g. [19, p.38]) to the function $f(z)e^{-i(a-\epsilon)z}$ in these regions, we conclude that it is bounded in $\{z \in \mathbb{C} : \ \operatorname{Im} z \geq H\}$ and therefore it is bounded in $\mathbb{C}_+$. $\square$

Chapter 5

Generalized Poisson representation of a function harmonic in the upper half-plane

5.1 A weakened version of the main result on representation of a harmonic function by a generalized Poisson integral

Theorem 5.1 *Let u be a function harmonic in $\mathbb{C}_+$ and satisfying the following two conditions:*

(i) *There exists a sequence $\{r_k\},\ r_k \to \infty$, such that*

$$\int_0^\pi u^+(re^{i\theta}) \sin\theta d\theta \le \exp\{o(r)\},\ r = r_k \to \infty. \tag{5.1}$$

(ii) *There exist $H > 0$ and $\alpha > 0$ such that*

$$\sup_{0<s<H} \int_{-\infty}^{\infty} \frac{|u(t+is)|}{1+|t|^\alpha} dt < \infty.$$

Then the assertion of Theorem 2.1 holds.

Proof. By Lemma 4.3, u admits representation

$$u(z) = \int_{-\infty}^{\infty} P_q(z,t)d\nu(t) + U(z), \tag{5.2}$$

where q and ν are as in Theorem 2.1 and U is a function harmonic in $\mathbb{C}$ such that $U(x) = 0,\ x \in \mathbb{R}$. Our aim is to show $U(z) = \operatorname{Im}\{P(z)\}$ where P is a real polynomial of degree at most q. The proof of this assertion is obtained in several steps.

1. Let us show that

$$\int_0^{\pi} U^+(re^{i\theta}) \sin\theta d\theta \le \exp(o(r)), \quad r = r_k \to \infty. \tag{5.3}$$

From (5.2) we get

$$\int_0^{\pi} U^+(re^{i\theta}) \sin\theta d\theta \le \int_0^{\pi} u^+(re^{i\theta}) \sin\theta d\theta + \int_0^{\pi} \sin\theta \left| \int_{-\infty}^{\infty} P_q(re^{i\theta}, t)\nu(t) \right| d\theta.$$

The first integral on the right hand side admits the estimate (5.1) and the second integral on the right hand side is $O(r^q)$ by Lemma 4.2. Thus, (5.3) holds.

2. Now we show that

$$\int_0^{\pi} |U(re^{i\theta})| \sin\theta d\theta \le \exp(o(r)), \quad r = r_k \to \infty. \tag{5.4}$$

By the Nevanlinna formula (see, Theorem N, p.26), taking into account that $U(t) = 0$ for $t \in \mathbb{R}$, we have

$$U(i) = \frac{1}{2\pi} \int_0^{\pi} \frac{4(r_k^2 - 1) r_k \sin\theta}{|r_k e^{i\theta} - i|^2 |r_k e^{i\theta} + i|^2} U(r_k e^{i\theta}) d\theta.$$

Note that for $r_k \ge 2,\ 0 \le \theta \le \pi$,

$$\frac{C_1}{r_k} \sin\theta \le \frac{4(r_k^2 - 1) r_k \sin\theta}{|r_k e^{i\theta} - i|^2 |r_k e^{i\theta} + i|^2} \le \frac{C_2}{r_k} \sin\theta,$$

where C_1 and C_2 are positive constants. Since $|U| = 2U^+ - U$, we obtain

$$\int_0^\pi |U(r_k e^{i\theta})| \sin\theta d\theta \leq \frac{r_k}{C_1} \int_0^\pi \frac{4(r_k^2-1) r_k \sin\theta}{|r_k e^{i\theta} - i|^2 |r_k e^{i\theta} + i|^2} |U(r_k e^{i\theta})| d\theta$$
$$\leq \frac{2C_2}{C_1} \int_0^\pi U^+(r_k e^{i\theta}) \sin\theta d\theta - \frac{2\pi r_k}{C_1} U(i).$$

Then (5.4) follows from (5.3).

3. Let us show that

$$|U(z)| \leq \exp(o(|z|)), \quad |z| = \frac{r_k}{2} \to \infty. \tag{5.5}$$

Since $U(z) = -U(\bar{z})$, $z \in \mathbb{C}$, it is enough to prove (5.5) only for $z \in \mathbb{C}_+$. Applying Nevanlinna formula once more, we get

$$U(z) = \frac{1}{2\pi} \int_0^\pi \frac{\left(r_k^2 - \frac{r_k^2}{4}\right) 4 r_k \frac{r_k}{2} \sin\theta \sin\varphi}{|r_k e^{i\theta} - \frac{r_k}{2} e^{i\varphi}|^2 |r_k e^{i\theta} - \frac{r_k}{2} e^{-i\varphi}|^2} U(r_k e^{i\theta}) d\theta, \ z = \frac{r_k}{2} e^{i\varphi} \in \mathbb{C}_+.$$

Simple estimates and (5.4) show that

$$|U(z)| \leq \frac{12}{\pi} \int_0^\pi |U(r_k e^{i\theta})| \sin\theta d\theta \leq \exp(o(r_k)).$$

4. Now, let us show that

$$|U(z)| = o(|z|^{q+1}), \quad z \to \infty, \ |\operatorname{Im} z| < \frac{H}{2}. \tag{5.6}$$

Formula (5.2) implies that U has a similar representation as the function U_s in formula (4.18). Calculations similar to (4.20) - (4.23) show that there exists a constant $C_{q,H} \geq 0$ such that

$$\int_{-\infty}^{\infty} \frac{|U(x+iy)|}{1+|x|^{q+1}} dx \leq C_{q,H}, \quad |y| \leq H, \tag{5.7}$$

instead of (4.19), and

$$|U(z)| \leq \frac{1 + (|z|+\rho)^{q+1}}{\pi \rho^2} \int_{\operatorname{Im} z - \rho}^{\operatorname{Im} z + \rho} \left\{ \int_{\operatorname{Re} z - \rho}^{\operatorname{Re} z + \rho} \frac{|U(\xi + i\eta)|}{1 + |\xi|^{q+1}} d\xi \right\} d\eta,$$

instead of (4.24). Putting $\rho = H/2$, we obtain for $|\operatorname{Im} z| < H/2$

$$|U(z)| \leq \frac{1 + (|z| + H/2)^{q+1}}{\pi (H/2)^2} \int\int \frac{|U(\xi + i\eta)|}{1 + |\xi|^{q+1}} d\xi d\eta,$$

where the double integral is taken over the rectangle

$$\left\{(\xi,\eta)\in\mathbb{R}^2:\ |\xi-\operatorname{Re}z|<\frac{H}{2},\ |\eta|<H\right\}.$$

This integral tends to 0 as $z\to\infty$ because its integrand is summable over the whole strip $\{(\xi,\eta)\in\mathbb{R}^2:\ |\eta|<H\}$ as seen in (5.7). Thus, (5.6) is valid.

5. Let G be the entire function which is determined uniquely by the conditions $\operatorname{Re}G(z)=U(z),\ G(0)=0$. We shall show that there exists a sequence $\{R_k\},\ R_k\to\infty$ such that

$$|G(z)|\le\exp\{o(|z|)\},\quad |z|=R_k\to\infty \tag{5.8}$$

and that

$$|G(z)|=o(|z|^{q+2}),\quad |\operatorname{Im}z|\le H/4,\ z\to\infty. \tag{5.9}$$

To this end we use the Schwarz formula

$$G(z+\zeta)=\frac{1}{2\pi}\int_0^{2\pi}U(z+\rho e^{i\theta})\frac{\rho e^{i\theta}+\zeta}{\rho e^{i\theta}-\zeta}d\theta+i\ \operatorname{Im}\{G(z)\},\quad |\zeta|<\rho.$$

Differentiating with respect to ζ and putting $\zeta=0$, we get

$$G'(z)=\frac{1}{\pi\rho}\int_0^{2\pi}U(z+\rho e^{i\theta})e^{-i\theta}d\theta.$$

This implies

$$|G'(z)|\le\frac{2}{\rho}\max_{|\zeta-z|\le\rho}|U(\zeta)|. \tag{5.10}$$

Now, choose $\rho=H/4$. For $|z|\le R_k:=r_k/2-\rho$, we get from (5.10) and (5.5)

$$\begin{aligned}|G(z)|&=\left|\int_0^z G'(\zeta)d\zeta\right|\le R_k\max_{|z|\le R_k}|G'(z)|\\&\le R_k\exp\left(o\left(\frac{r_k}{2}\right)\right)=\exp(o(R_k)),\quad R_k\to\infty.\end{aligned}$$

Hence, (5.8) is valid.

For $|\operatorname{Im}z|\le H/4$, we get from (5.10) and (5.6)

$$|G'(z)|=o(|z|^{q+1}),\quad z\to\infty,$$

whence (5.9) follows by integration.

6. Let us complete the proof of Theorem 5.1.

We apply the well-known version of the Phragmén-Lindelöf principle for half-plane (see, e.g. [22, p.43]) to the function $G(z)/(z+i)^{q+2}$ in $\mathbb{C}_+$ and to the function $G(z)/(z-i)^{q+2}$ in $\mathbb{C}_-$. This shows that (5.9) holds in the whole complex plane $\mathbb{C}$. But then, by Liouville's theorem, the function G is a polynomial of degree at most $q+1$. Since $\operatorname{Re} G(t) = U(t) = 0,\ t \in \mathbb{R}$ and $G(0) = 0$, we have $G(z) = ia_{q+1}z^{q+1} + ia_q z^q + \cdots + ia_1 z,\ a_j \in \mathbb{R},\ j = 1, 2, \cdots, q+1$. Hence $U(z) = \operatorname{Im}\{-a_{q+1}z^{q+1} - \cdots - a_1 z\}$. Clearly, (5.7) yields $a_{q+1} = 0$, so that (2.1) holds. □

5.2 A local representation of a harmonic function by a generalized Poisson kernel

Proof of Lemma 2.3. We proceed in a manner similar to the proof of Lemma 4.3.

We fix R and consider the following family of Borel measures on $[-R, R]$:

$$\nu_{R,s}(E) = \int_E u(t+is)dt, \quad E \subset [-R, R], \quad 0 < s < H.$$

Each sequence $\{\nu_{R,s_k}\}$, $\lim_{k\to\infty} s_k = 0$, contains a subsequence (which we also denote by $\{\nu_{R,s_k}\}$) which is weak-star convergent to a finite Borel measure ν_R on $[-R, R]$. Hence,

$$\lim_{k\to\infty} \int_{-R}^{R} u(t+is_k)P_q(z,t)dt = \int_{-R}^{R} P_q(z,t)d\nu_R(t). \tag{5.11}$$

Consider the following family of functions

$$U_{R,s}(z) = u(z+is) - \int_{-R}^{R} P_q(z,t)u(t+is)dt, \quad z \in \mathbb{C}_+, \quad 0 < s < \frac{H}{2}. \tag{5.12}$$

Clearly, $U_{R,s}$ is harmonic in $\mathbb{C}_+$ and continuous in $\mathbb{C}_+ \cup (-R, R)$ if we define $U_{R,s}(x) = 0$ for $x \in \mathbb{R}$. By the Symmetry Principle, $U_{R,s}$ can be extended to a function (which we also denote by $U_{R,s}$) harmonic in $\mathbb{C}\setminus[(-\infty, -R] \cup [R, \infty)]$ and satisfies $U_{R,s}(z) = -U_{R,s}(\bar{z})$ there.

Take any $R^{'} > R$, any $\epsilon > 0$ sufficiently small and consider the compact set

$$\Pi_{R,R',\epsilon} := \left\{ z \in \mathbb{C} : \ |\operatorname{Re} z| \le R^{'}, \ |\operatorname{Im} z| \le \frac{H}{4} \right\} \\ \setminus \{ z \in \mathbb{C} : \ |\operatorname{Re} z| > R - \epsilon, \ |\operatorname{Im} z| < \epsilon \}$$

(see, Figure 1).

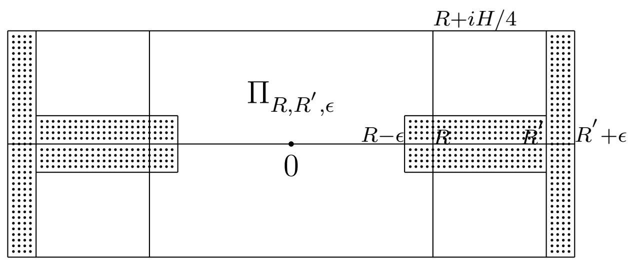

FIGURE 1

We shall show that the family $\{U_{R,s} : \ 0 < s < H/2\}$ is uniformly bounded on the set $\Pi_{R,R',\epsilon}$.

To do so, we first prove that

$$\sup_{0<y<H/2} \int_{-(R'+\epsilon)}^{(R'+\epsilon)} |U_{R,s}(x+iy)|dx < \infty. \tag{5.13}$$

It follows from (5.12) that

$$\int_{-(R'+\epsilon)}^{R'+\epsilon} |U_{R,s}(x+iy)|dx \le \int_{-(R'+\epsilon)}^{R'+\epsilon} |u(x+i(y+s))|dx \\ + \int_{-(R'+\epsilon)}^{R'+\epsilon} \left\{ \int_{-R}^{R} |P_q(x+iy,t)||u(t+is)|dt \right\} dx =: I_1 + I_2.$$

Using Fubini's theorem and estimate (3.4) we get

$$\begin{aligned} I_2 &= \int_{-R}^{R} |u(t+is)| \left\{ \int_{-(R'+\epsilon)}^{R'+\epsilon} |P_q(x+iy,t)|dx \right\} dt \\ &\le C_{q,R'} \int_{-R}^{R} |u(t+is)| \left\{ \int_{-(R'+\epsilon)}^{R'+\epsilon} \frac{y}{(x-t)^2+y^2} dx \right\} dt \\ &\le C_{q,R'} \int_{-R}^{R} |u(t+is)|dt \end{aligned}$$

where $C_{q,R'}$ does not depend on y and s. Since $0 < s < H/2$, it follows from (2.10) that I_2 is bounded by a constant which does not depend on y or s.

For $0 < y < H/2$, we have $0 < y+s < H$ and (2.10) shows that I_1 is uniformly bounded. Hence (5.13) holds.

Since $|U_{R,s}|$ is subharmonic in $\mathbb{C}\setminus[(-\infty,-R]\cup[R,\infty)]$, for $z \in \Pi_{R,R',\epsilon}$ we obtain the estimate

$$\begin{aligned}|U_{R,s}(z)| &\leq \frac{1}{\pi\epsilon^2}\int\int_{|\xi+i\eta-z|\leq\epsilon}|U_{R,s}(\xi+i\eta)|d\xi d\eta\\ &\leq \frac{1}{\pi\epsilon^2}\int_{-H/2}^{H/2}\int_{-(R'+\epsilon)}^{R'+\epsilon}|U_{R,s}(\xi+i\eta)|d\xi d\eta.\end{aligned}$$

By the aid of (5.13) this shows that the family $\{U_{R,s}:\ 0 < s < H/2\}$ is uniformly bounded on $\Pi_{R,R',\epsilon}$.

Now, let $\{s_k\}$ be a sequence such that (5.11) holds. By the Compactness Principle for harmonic functions (see, Theorem P, p.26), we can extract a subsequence (which we also denote by $\{s_k\}$) such that the sequence $\{U_{R,s_k}\}$ is uniformly convergent on any compact subset of the slit strip

$$\Pi = \left\{z \in \mathbb{C}:\ |\operatorname{Im} z| < \frac{H}{4}\right\}\setminus[(-\infty,-R]\cup[R,\infty)].$$

Let U_R be the limiting function. Clearly, U_R is harmonic in Π and satisfies $U(x) = 0,\ x \in (-R,R)$. With the choice $s = s_k$ in (5.12) and taking limit as $k \to \infty$, we obtain

$$U_R(z) = u(z) - \int_{-R}^{R} P_q(z,t)d\nu_R(t) \quad \text{for} \quad 0 < \operatorname{Im} z < \frac{H}{4}. \tag{5.14}$$

The right hand side of (5.14) is a harmonic function in $\mathbb{C}_+$. Therefore U_R can be extended harmonically to $\mathbb{C}_+$. Since $U_R(x) = 0,\ x \in (-R,R)$, the function U_R can further be extended harmonically to $\mathbb{C}\setminus[(-\infty,-R]\cup[R,\infty)]$.

Now let us show that for $R_2 > R_1$, the restriction of ν_{R_2} to $[-R_1,R_1]$ coincides with ν_{R_1}.

We have

$$U_{R_2}(z) - U_{R_1}(z) + \int_{R_1<|t|\leq R_2} P_q(z,t)d\nu_{R_2}(t) = \int_{-R_1}^{R_1} P_q(z,t)[d\nu_{R_1}(t) - d\nu_{R_2}(t)].$$

The left-hand side is a harmonic function in $\mathbb{C}\setminus[(-\infty,-R_1]\cup[R_1,\infty)]$ vanishing on $(-R_1,R_1)$. Since

$$P_q(z,t)=\frac{y}{\pi}\frac{1}{(x-t)^2+y^2}+Q(x,y,t), \tag{5.15}$$

where Q is a harmonic polynomial in x and y vanishing for $y=0$ (see, (3.8), arguments on p.19 and Theorem G), we see that ν_{R_2} coincides with ν_{R_1} on $(-R_1,R_1)$. $\square$

5.3 Harmonic functions with growth restrictions on two horizontal lines

Proof of Theorem 2.4. Let ν be the σ-finite Borel measure defined in Lemma 2.3 and $q=\max\{n\in\mathbb{N}\cup\{0\}:\ n<\alpha\}$. Since $\alpha\le q+1$, it follows from (2.11) that

$$\int_{-\infty}^{\infty}\frac{d|\nu|(t)}{1+|t|^{q+1}}<\infty,$$

and hence by Theorem 3.3

$$\int_{-\infty}^{\infty}P_q(z,t)d\nu(t).$$

represents a function harmonic in $\mathbb{C}_+$.

Define

$$U(z)=u(z)-\int_{-\infty}^{\infty}P_q(z,t)d\nu(t).$$

We shall show that $U(z)=\operatorname{Im}P(z)$ with some real polynomial P of degree at most q.

For any $R>0$, we write

$$U(z)=\left\{u(z)-\int_{-R}^{R}P_q(z,t)d\nu(t)\right\}-\int_{|t|>R}P_q(z,t)d\nu(t)=:I_1+I_2.$$

It is easy to see from Lemma 2.3 that I_1 is a harmonic function in the slit plane $\mathbb{C}\setminus[(-\infty,-R]\cup[R,\infty)]$ vanishing on $(-R,R)$. Using (5.15), we see that I_2 has this property, too. Hence U is continuous in $\overline{\mathbb{C}_+}$, if we set $U(x)=0,\ x\in\mathbb{R}$.

By an application of the Symmetry Principle, we can extend U to a harmonic function in $\mathbb{C}$ (which we also denote by U).

Repeating steps 1-3 of the proof of Theorem 5.1, we obtain

$$|U(re^{i\theta})| \le \exp(o(r)), \quad r = \frac{r_k}{2} \to \infty. \tag{5.16}$$

Suppose that we have shown

$$|U(z)| = O(|z|^{q+3}), \quad z \to \infty, \quad |\operatorname{Im} z| < \frac{H}{2}. \tag{5.17}$$

Then using the methods of steps 5 and 6 in the proof of Theorem 5.1, we get that $U = \operatorname{Im}\{P(z)\}$, where P is a real polynomial of degree at most $q+4$. Noting that the convergence of the first integral in (2.11) implies

$$\int_{-\infty}^{\infty} \frac{|U(x+iH)|}{1+|x|^{q+1}} dx < \infty, \tag{5.18}$$

we obtain that in fact degree of P is not greater than q and hence the assertion of Theorem 2.4 holds. Now it only remains to establish (5.17).

Set

$$v(z) = U(z) - \int_{-\infty}^{\infty} P_q(x+i(H-y),t)U(t+iH)dt,\ z = x+iy,\ y < H. \tag{5.19}$$

By (5.18) and Theorem 3.3, v is harmonic in $\{z \in \mathbb{C} :\ \operatorname{Im} z < H\}$ and continuous in $\{z \in \mathbb{C} :\ \operatorname{Im} z \le H\}$, if we set $v(x+iH) = 0,\ x \in \mathbb{R}$. Hence v can be extended to a harmonic function in $\mathbb{C}$ (which we also denote by v).

Now, we show that there exists a sequence $\{R_k\},\ R_k \to \infty$ such that

$$|v(re^{i\theta})| \le \exp(o(r)),\ r = R_k \to \infty. \tag{5.20}$$

Set $\tilde{v}(\zeta) := v(\zeta + iH)$. Then, by (5.19), with $\rho_k = r_k/2 - H$,

$$\begin{aligned}\int_{-\pi}^{0} |\tilde{v}(\rho_k e^{i\theta})||\sin\theta|d\theta \le & \int_{-\pi}^{0} |U(\rho_k e^{i\theta} + iH)||\sin\theta|d\theta \\ & + \int_{-\pi}^{0} \left\{ \int_{-\infty}^{\infty} |P_q(\rho_k e^{-i\theta}, t)||U(t+iH)|dt \right\} |\sin\theta|d\theta \\ =: & I_1 + I_2.\end{aligned}$$

From (5.16), we obtain $I_1 \leq \exp(o(\rho_k)),\ \rho_k \to \infty$. Using Fubini's Theorem and the estimate (3.4), we get $I_2 \leq \exp(o(\rho_k)),\ \rho_k \to \infty$ by an argument similar to that at the end of the first step in the proof of Theorem 5.1. By symmetry,

$$\int_0^{\pi} |\tilde{v}(\rho_k e^{i\theta})| \sin\theta d\theta \leq \exp(o(\rho_k)), \quad \rho_k \to \infty. \tag{5.21}$$

In the same fashion as in the third step of the proof of Theorem 5.1, we obtain from (5.21) that

$$|\tilde{v}(z)| \leq \exp(o(|z|)), \quad |z| = \frac{\rho_k}{2} \to \infty.$$

Since $|v|$ is a subharmonic function in $\mathbb{C}$, (5.20) holds for $R_k = \rho_k/2 - H$.

Now, we show that

$$|v(x)| \leq O(|x|^{q+2}), \quad x \to \infty, \quad x \in \mathbb{R}. \tag{5.22}$$

Since $U(x) = 0,\ x \in \mathbb{R}$, it follows from (5.19) and estimate (3.4) that for $|x|$ sufficiently large

$$\begin{aligned}
|v(x)| &\leq \int_{-\infty}^{\infty} |P_q(x+iH,t)||U(t+iH)|dt \\
&\leq D_q|x|^q \int_{-\infty}^{\infty} \frac{H}{(x-t)^2+H^2} \cdot \frac{1}{(1+|t|)^{q-1}} |U(t+iH)|dt \\
&\leq D_{q,H}|x|^q \left\{ \int_{|t|<1} |U(t+iH)|dt + \int_{|t|\geq 1} \frac{|U(t+iH)|}{1+|t|^{q+1}} \cdot \frac{2t^2}{(x-t)^2+H^2} dt \right\} \\
&= O(|x|^q) + D_q|x|^q \max_{|t|\geq 1} \frac{2t^2}{(x-t)^2+H^2} \int_{|t|\geq 1} \frac{|U(t+iH)|}{1+|t|^{q+1}} dt \\
&= O(|x|^{q+2}),\ |x| \to \infty.
\end{aligned}$$

Therefore, (5.22) is true.

Now, let V be an entire function such that $\operatorname{Re} V(z) = v(z)$. Define

$$F(z) := \begin{cases} e^{V(z)-Az^{q+2}} & \text{if } q \text{ is even}, \\ e^{V(z)-Az^{q+3}} & \text{if } q \text{ is odd}, \end{cases} \tag{5.23}$$

where $A > 0$ is a constant. It is evident from (5.22) that F is bounded on the boundary of the strip $S := \{z \in \mathbb{C} :\ 0 < \operatorname{Im} z < H\}$ for a large enough A.

From (5.20), it follows that

$$|F(z)| \leq \exp\exp(o(|z|)),\ |z| = R_k \to \infty,\ z \in S.$$

Applying Phragmén-Lindelöf principle for strip (see, e.g. [19, p.40]), we conclude that F is bounded on S. From this it follows that

$$v(z) = \log |F(z)| + O(|z|^{q+3}) \leq O(|z|^{q+3}), \quad z \to \infty,\ z \in S. \tag{5.24}$$

Similarly, replacing V by $-V$ in (5.23), we get

$$-v(z) \leq O(|z|^{q+3}), \quad z \to \infty,\ z \in S. \tag{5.25}$$

From (5.24) and (5.25) we conclude

$$|v(z)| = O(|z|^{q+3}), \quad z \to \infty,\ z \in S. \tag{5.26}$$

As in proof of (5.22), we show that

$$\left| \int_{-\infty}^{\infty} P_q(x + i(H-y), t) U(t + iH) dt \right| = O(|z|^{q+2}),\ z \to \infty,\ 0 < \operatorname{Im} z < H/2. \tag{5.27}$$

Now (5.17) follows from (5.19), (5.26) and (5.27). □

5.4 Main result on representation of harmonic functions by generalized Poisson integrals

Proof of Theorem 2.1. To prove Theorem 2.1, let us note that it suffices to show that its conditions imply condition (4.13). Then assertion of Theorem 2.1 will follow from Theorem 5.1.

By condition (2.6) of Theorem 2.1, there exists a constant $C > 0$ and a decreasing sequence $\{s_l\}_{l=1}^{\infty} \subset (0, 1]$, $\lim_{l\to\infty} s_l = 0$, such that

$$\int_{-\infty}^{\infty} \frac{|u(x + is_l)|}{1 + |x|^{\alpha}} dx \leq C,\ l = 1, 2, \cdots, \tag{5.28}$$

holds. Clearly, it suffices to show that

$$\sup_{s_l < y < s_1} \int_{-\infty}^{\infty} \frac{|u(x + iy)|}{1 + |x|^{\alpha}} \leq D,\ l = 1, 2, \cdots, \tag{5.29}$$

where D does not depend on l.

We want to apply Lemma 3.4 to $v_l(z) = u(z+is_l)$ for $h = h_l := s_1 - s_l,\ l > 1$. It is clear from (5.28) that v_{l+1} satisfies (3.10) of Lemma 3.4. To show that (3.9) of Lemma 3.4 is also satisfied, we need to apply Corollary 2.5 of Theorem 2.4 to the function $v_{l+1}(z) = u(z + is_{l+1})$. It is easy to see that condition (2.12) is satisfied with $H = h_{l+1}$. Now, we need to show that there exists a sequence $\{R_k\},\ R_k \to \infty$ such that

$$\int_0^\pi v_{l+1}^+(re^{i\theta}) \sin\theta d\theta \le \exp(o(r)),\ r = R_k \to \infty.$$

Let

$$Q_{R,s_{l+1}} = \{z \in \mathbb{C}:\ |z + is_{l+1}| < R\} \cap \mathbb{C}_+,\ R > s_{l+1}.$$

Since the function $v_{l+1}^+(z)$ is subharmonic in the closure of $Q_{R,s_{l+1}}$, we have

$$v_{l+1}^+(z) \le \frac{1}{2\pi} \int_{\partial Q_{R,s_{l+1}}} v_{l+1}^+(\zeta) \frac{\partial G_{Q_{R,s_{l+1}}}}{\partial n}(\zeta, z)|d\zeta|, \quad z \in Q_{R,s_{l+1}},$$

where $G_{Q_{R,s_{l+1}}}(\zeta, z)$ is the Green function of $Q_{R,s_{l+1}}$ and $\partial/\partial n$ is derivative in the direction of the inner normal.

Let

$$K_{R,s_{l+1}} = \{z \in \mathbb{C}:\ |z + is_{l+1}| < R,\ \operatorname{Im} z > -s_{l+1}\}.$$

According to the Principle of Extension of Domains (see, [4, p.3], or in an equivalent form [22, Ch. IV §2]), we have

$$\frac{\partial G_{Q_{R,s_{l+1}}}}{\partial n}(\zeta, z) \le \frac{\partial G_{K_{R,s_{l+1}}}}{\partial n}(\zeta, z),\ \text{for}\ \zeta \in \partial Q_{R,s_{l+1}} \cap \partial K_{R,s_{l+1}},\ z \in Q_{R,s_{l+1}}.$$

Using the well-known explicit expression for the Green function of a half-disc, we get ($z_{s_{l+1}} = z + is_{l+1},\ \varphi_{s_{l+1}} = \arg z_{s_{l+1}}$)

$$\begin{aligned}
\int_{\partial Q_{R,s_{l+1}}\cap\partial K_{R,s_{l+1}}} & v_{l+1}^+(\zeta)\frac{\partial G_{Q_{R,s_{l+1}}}}{\partial n}(\zeta,z)|d\zeta| \\
&\le \int_{\partial Q_{R,s_{l+1}}\cap\partial K_{R,s_{l+1}}} v_{l+1}^+(\zeta)\frac{\partial G_{K_{R,s_{l+1}}}}{\partial n}(\zeta,z)|d\zeta| \\
&= \int_{\arcsin(s_{l+1}/R)}^{\pi-\arcsin(s_{l+1}/R)} u^+(Re^{i\theta})\frac{4R|z_{s_{l+1}}|(R^2-|z_{s_{l+1}}|^2)\sin\theta\sin\varphi_{s_{l+1}}d\theta}{|Re^{i\theta}-z_{s_{l+1}}|^2|Re^{-i\theta}-z_{s_{l+1}}|^2} \\
&\le \frac{4(R+|z|+s_{l+1})^3}{(R-|z|-s_{l+1})^3}\int_0^\pi u^+(Re^{i\theta})\sin\theta d\theta. \qquad (5.30)
\end{aligned}$$

Further, since $Q_{R,s_{l+1}} \subset \mathbb{C}_+$, we have

$$\frac{\partial G_{Q_{R,s_{l+1}}}}{\partial n}(\zeta,z) \le \frac{\partial G_{\mathbb{C}_+}}{\partial n}(\zeta,z),\ \text{for } \zeta\in\partial Q_{R,s_{l+1}}\backslash\mathbb{C}_+,\ z\in Q_{R,s_{l+1}}.$$

Hence

$$\begin{aligned}
\int_{\partial Q_{R,s_{l+1}}\backslash\mathbb{C}_+} & v_{l+1}^+(\zeta)\frac{\partial G_{Q_{R,s_{l+1}}}}{\partial n}(\zeta,z)|d\zeta| \\
&\le \int_{\partial Q_{R,s_{l+1}}\backslash\mathbb{C}_+} v_{l+1}^+(\zeta)\frac{\partial G_{\mathbb{C}_+}}{\partial n}(\zeta,z)|d\zeta| \\
&= \int_{-\sqrt{R^2-s_{l+1}^2}}^{\sqrt{R^2-s_{l+1}^2}} u^+(t+is_{l+1})\frac{2y}{(x-t)^2+y^2}dt \\
&\le \int_{-R}^{R} u^+(t+is_{l+1})\frac{2ydt}{(x-t)^2+y^2},\quad z=x+iy\in Q_{R,s_{l+1}}. \qquad (5.31)
\end{aligned}$$

Joining (5.30) and (5.31), we obtain

$$\begin{aligned}
v_{l+1}^+(re^{i\varphi}) \le & \frac{2(R+r+s_{l+1})^3}{\pi(R-r-s_{l+1})^3}\int_0^\pi u^+(Re^{i\theta})\sin\theta d\theta \\
& + \frac{1}{\pi}\int_{-R}^{R} u^+(t+is_{l+1})\frac{r\sin\varphi dt}{r^2+t^2-2rt\cos\varphi},\ re^{i\varphi}\in Q_{R,s_{l+1}}. \qquad (5.32)
\end{aligned}$$

For $0<r<R-s_{l+1}$, we have $re^{i\varphi}\in Q_{R,s_{l+1}}$ when $0<\varphi<\pi$. Let us multiply both sides of (5.32) by $\sin\varphi$ and integrate with respect to φ from 0 to π. Using the relation (4.12), we obtain

$$\int_0^\pi v_{l+1}^+(re^{i\varphi})\sin\varphi d\varphi \le \frac{4(R+r+s_{l+1})^3}{\pi(R-r-s_{l+1})^3}\int_0^\pi u^+(Re^{i\theta})\sin\theta d\theta \\ + C_q(1+R)^{\alpha-2} r\int_{-\infty}^{\infty}\frac{u^+(t+is_{l+1})dt}{1+|t|^\alpha}. \quad (5.33)$$

Remind that function u satisfies (2.5). Set $R = r_k,\ r = r_k/2 - s_{l+1}$ in (5.33) where $\{r_k\}$ is the sequence from (2.5). Then we see that condition (2.5) is satisfied for $v_{l+1}^+(z)$ with $\{R_k\} = \{r_k/2 - s_{l+1}\}$ instead of $\{r_k\}$.

Now, by Corollary 2.5, v_{l+1} admits the representation

$$v_{l+1}(z) = \int_{-\infty}^{\infty} P_q(z,t)u(t+is_{l+1})dt + \mathrm{Im}\{P_{l+1}(z)\}, \quad (5.34)$$

where $q = \max\{n \in \mathbb{N}\cup\{0\},\ n < \alpha\}$ and P_{l+1} is a polynomial of degree at most q. Using representation (5.34) and estimate (3.4), we get

$$|v_l(x+iy)| = |v_{l+1}(x+iy+i(s_l-s_{l+1}))| \\ \le C_q(1+|x|+h_{l+1})^q\int_{-\infty}^{\infty}\frac{|u(t+is_{l+1})|}{(1+|t|)^{q-1}}\cdot\frac{(y+(s_l-s_{l+1}))dt}{(x-t)^2+(y+(s_l-s_{l+1}))^2} \\ + O(|x|^q),\ |x|\to\infty,\quad 0<y<h_l.$$

As in proof of (5.22), this shows that

$$|v_l(x+iy)| = O(|x|^{q+2}),\ |x|\to\infty,\ 0<y<h_l,$$

and hence v_{l+1} satisfies condition (3.9) of Lemma 3.4. Applying Lemma 3.4, we get for $v_l(z)$ the formula (3.11). Dividing both its sides by $1+|x|^\alpha$, integrating with respect to x and changing the order of integration, we obtain

$$\int_{-\infty}^{\infty}\frac{|u(x+i(s_l+y))|}{1+|x|^\alpha}dx \le \int_{-\infty}^{\infty}|u(t+is_l)|\cdot I_l^+(t)dt \\ + \int_{-\infty}^{\infty}|u(t+is_1)|\cdot I_l^-(t)dt,\quad 0<y<h_l. \quad (5.35)$$

where

$$I_l^\pm(t) = \frac{\sin(\pi y/h_l)}{2h_l}\int_{-\infty}^{\infty}\frac{dx}{(\cosh[\pi(x-t)/h_l]\pm\cos(\pi y/h_l))(1+|x|^\alpha)}.$$

Let us estimate $I_l^{\pm}(t)$:

$$I_l^{\pm}(t) = \frac{\sin(\pi y/h_l)}{2h_l}\int_{|x|>|t|/2} + \frac{\sin(\pi y/h_l)}{2h_l}\int_{|x|<|t|/2} =: I_{1,l}^{\pm}(t) + I_{2,l}^{\pm}(t).$$

Evidently,

$$I_{1,l}^{\pm}(t) \le \frac{1}{1+|t/2|^{\alpha}} \cdot \frac{\sin(\pi y/h_l)}{2h_l}\int_{-\infty}^{\infty}\frac{dx}{\cosh[\pi(x-t)/h_l] \pm \cos(\pi y/h_l)};$$

$$I_{2,l}^{\pm}(t) \le \frac{1}{2h_l}\int_{|x|<|t|/2}\frac{dx}{\cosh[\pi(x-t)/h_l]-1} \le \frac{1}{2h_l}\frac{|t|/2}{\cosh(\pi t/(2h_l))-1}.$$

Choosing $n \in \mathbb{N}$ in such a way that $2n > \alpha+1$ and using the inequality $\cosh\tau - 1 \ge \tau^{2n}/(2n)!$, we get

$$I_{2,l}^{\pm}(t) \le \frac{|t|}{4h_l}\cdot\frac{(2n)!(2h_l)^{2n}}{(\pi|t|)^{2n}} \le \frac{D_1}{1+|t|^{\alpha}}, \quad \text{for } |t| \ge 1, \tag{5.36}$$

where D_1 does not depend on l (remind that $h_l < 1$).

Note that by Lemma 3.4 for $v(z) \equiv 1$, we have the identity

$$1 = \frac{\sin(\pi y/h_l)}{2h_l}\left\{\int_{-\infty}^{\infty}\frac{dx}{\cosh[\pi(x-t)/h_l]-\cos(\pi y/h_l)} + \int_{-\infty}^{\infty}\frac{dx}{\cosh[\pi(x-t)/h_l]+\cos(\pi y/h_l)}\right\},$$

and both integrals in the right hand side are positive. Hence we conclude that

$$I_{1,l}^{\pm}(t) \le \frac{D_2}{1+|t|^{\alpha}}, \quad t \in \mathbb{R},$$

and, moreover,

$$I_l^{\pm}(t) = I_{1,l}^{\pm}(t) + I_{2,l}^{\pm}(t) \le 1, \quad t \in \mathbb{R}.$$

Using this and (5.36), we conclude that

$$I_l^{\pm}(t) \le \frac{D_3}{1+|t|^{\alpha}}, \quad t \in \mathbb{R},$$

where D_3 does not depend on l and t.

Substituting this estimate into (5.35), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{|u(x+is_l+iy)|}{1+|x|^{\alpha}}dx &\le D_3\int_{-\infty}^{\infty}\frac{|u(t+is_l)|+|u(t+is_1)|}{1+|t|^{\alpha}}dt \\ &\le 2D_3C, \quad 0<y<s_1-s_l,\end{aligned}$$

where C is the constant from (5.28), and (5.29) follows. □

Chapter 6

Applications to the Hardy and the Nevanlinna classes

6.1 Factorization in the Nevanlinna class

Proof of Theorem 2.6. Since h belongs to the Nevanlinna class its zeros satisfy the Blaschke condition (see, Corollary 3.5, p.24). Therefore the zeros of g_1 and g_2 also satisfy this condition.

Denote by B_1 the Blaschke product corresponding to the zeros of g_1. The function

$$f := \frac{g_1}{B_1}$$

is analytic and non-vanishing in $\mathbb{C}_+$. So $\log|f|$ is harmonic in $\mathbb{C}_+$.

Let us show that $\log|f|$ satisfies the conditions of Corollary 2.2.

We have

$$\int_0^\pi \log^+|f(re^{i\theta})|\sin\theta d\theta \leq \int_0^\pi \log^+|g_1(re^{i\theta})|\sin\theta d\theta + \int_0^\pi \log^+\frac{1}{|B_1(re^{i\theta})|}\sin\theta d\theta.$$

Thus, it follows from (2.14) and Lemma 4.1 that $\log|f|$ satisfies condition (2.7) of Corollary 2.2.

Observe also that

$$\int_{-\infty}^{\infty} \frac{|\log|f(t+is)||}{1+t^2}dt \le \int_{-\infty}^{\infty} \frac{|\log|g_1(t+is)||}{1+t^2}dt + \int_{-\infty}^{\infty} \frac{\log^+|1/B_1(t+is)|}{1+t^2}dt.$$

Further

$$\int_{-\infty}^{\infty} \frac{|\log|g_1(t+is)||}{1+t^2}dt = \int_{-\infty}^{\infty} \frac{\log^+|g_1(t+is)|}{1+t^2}dt + \int_{-\infty}^{\infty} \frac{\log^+|1/g_1(t+is)|}{1+t^2}dt,$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\log^+|1/g_1(t+is)|}{1+t^2}dt &= \int_{-\infty}^{\infty} \frac{\log^+|g_2(t+is)/h(t+is)|}{1+t^2}dt \\ &\le \int_{-\infty}^{\infty} \frac{\log^+|g_2(t+is)|}{1+t^2}dt + \int_{-\infty}^{\infty} \frac{\log^+|1/h(t+is)|}{1+t^2}dt.\end{aligned}$$

So, combining all these together, we get

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{|\log|f(t+is)||}{1+t^2}dt &\le \int_{-\infty}^{\infty} \frac{\log^+|g_1(t+is)|}{1+t^2}dt + \int_{-\infty}^{\infty} \frac{\log^+|g_2(t+is)|}{1+t^2}dt \\ &+ \int_{-\infty}^{\infty} \frac{\log^+|1/h(t+is)|}{1+t^2}dt + \int_{-\infty}^{\infty} \frac{\log^+|1/B_1(t+is)|}{1+t^2}dt.\end{aligned}$$

Now, by condition (2.15) and Corollary 3.6 (see, p.25) we see that $\log|f|$ also satisfies condition (2.8) of Corollary 2.2.

Applying Corollary 2.2 to the function $\log|f|$, we get the following representation:

$$\log|f(z)| = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\nu(t)}{(x-t)^2+y^2} + cy, \quad z = x+iy \in \mathbb{C}_+,$$

where ν is a real-valued Borel measure satisfying

$$\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1+t^2} < \infty,$$

and c is a real constant.

Let us complete the proof of the theorem.

Taking into account that $|g_1| \le |f|$ for $z \in \mathbb{C}_+$, we obtain

$$\log |g_1| \le \log |f| \le \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\nu^+(t)}{(x-t)^2 + y^2} + c_+ y$$

where $c_+ = \max(c, 0)$.

Thus $\log |g_1|$ has positive harmonic majorant, that is, g_1 belongs to the Nevanlinna class.

Since h and g_1 are functions of the Nevanlinna class, by Theorem J (see, p.24) and its Corollary 3.5, we can write h and g_1 in the form

$$h = B \cdot \frac{H_1}{H_2} \quad g_1 = B_1 \frac{G_1}{G_2}$$

where B and B_1 are Blaschke products formed by zeros of h and g_1 respectively, H_j, G_j are analytic and non-vanishing in $\mathbb{C}_+$ such that $|H_j| < 1$, $|G_j| < 1$. Then g_2 is in the form

$$g_2 = \frac{h}{g_1} = \frac{B}{B_1} \cdot \frac{H_1 G_2}{H_2 G_1}.$$

Note that $B_2 := B/B_1$ is the Blaschke product formed by zeros of g_2 and therefore B_2 is analytic in $\mathbb{C}_+$ and $|B_2| < 1$. Putting $F_1 := B_2 H_1 G_2$ and $F_2 := H_2 G_1$, we write $g_2 = F_1/F_2$ where F_j's are analytic in $\mathbb{C}_+$ and $|F_j| < 1$, $j = 1, 2$, and $F_2 \neq 0$. Hence, by Theorem J (see, p.24) g_2 belongs to the Nevanlinna class. □

Proof of Corollary 2.7. Clearly h, g_1 and g_2 satisfy the conditions of Theorem 2.6. Then according to the Theorem 2.6, g_1 and g_2 belong to the Nevanlinna class. Thus Corollary 2.7 follows from Lemma 4.4. □

Now let us consider some examples related to the sharpness of Theorem 2.6.

Example 6.1. Consider the functions $g_1(z) = \exp(\cos z)$, $g_2(z) = \exp(-\cos z)$. Evidently, these functions satisfy condition (2.14) with "$O(r)$" instead of "$o(r)$", satisfy condition (2.15) and satisfy $g_1 g_2 \equiv 1 \in H^\infty(\mathbb{C}_+)$. Nevertheless, neither g_1 nor g_2 belongs to the Nevanlinna class. Indeed, if f is a function belonging to the Nevanlinna class, then by representation (4.27) and similar estimation as in the proof of Lemma 4.4 (cf. (4.28)), we have

$$\log^+ |f(z)| \leq ky + o(|z|), \quad |z| \to \infty, |\pi/2 - \arg z| \leq \pi/4. \tag{6.1}$$

But, in our case $\log^+ |g_1(iy)| = (e^y + e^{-y})/2$ does not satisfy the above inequality. This shows that g_1 does not belong to the Nevanlinna class. Hence g_2 does not belong to the Nevanlinna class. This example shows that the condition (2.14) cannot be weakened replacing $o(r)$ by $O(r)$.

The following example shows that the condition (2.15) in Theorem 2.1 cannot be weakened by replacing it with

$$\exists H > 0, \quad \sup_{0<s<H} \int_{-\infty}^{\infty} \frac{\log^+ |g_j(t+is)|}{1+|t|^\alpha} dt < \infty, \quad j = 1, 2,$$

for some $\alpha > 2$.

Example 6.2. Consider the functions $g_1(z) = \exp\{iz^2\}$, $g_2(z) = \exp\{-iz^2\}$. Since $\log |g_1(t+is)| = -2ts$, $\log |g_2(t+is)| = 2ts$, they satisfy the above condition. The condition (2.14) of Theorem 2.1 and $g_1 g_2 \equiv 1 \in H^\infty(\mathbb{C}_+)$ are also satisfied. Nevertheless, neither g_1 nor g_2 belongs to the Nevanlinna class by the same reason as in the Example 6.1, now it is enough to look the growth of $\log^+ |g_1|$ and $\log^+ |g_2|$ on the rays $\{z \in \mathbb{C} : \arg z = 3\pi/4\}$ and $\{z \in \mathbb{C} : \arg z = \pi/4\}$ respectively.

The condition (2.14) of Theorem 2.1 touches only one of the functions g_1, g_2. But it is impossible to change the condition (2.15) in a similar way as the following example shows.

Example 6.3. Consider the functions $g_1(z) = \exp\{z^2\}$, $g_2(z) = \exp\{-z^2\}$. Clearly g_1 and g_2 satisfy the condition (2.14) and satisfy $g_1 g_2 \equiv 1 \in H^\infty(\mathbb{C}_+)$ and g_2 satisfy condition (2.15) but for any $\epsilon > 0$

$$\sup_{0<s<\epsilon} \int_{\infty}^{\infty} \frac{\log^+ |g_1(t+is)|}{1+t^2} dt = \infty.$$

In this case, we see by the same reason as in the previous examples that neither g_1 nor g_2 belong to the Nevanlinna class.

6.2 Factorization in $H^{\infty}(\mathbb{C}_+)$ when the factors are connected by a linear equation

Proof of Theorem 2.8. We divide the proof into 4 steps.

1. Let us show that

$$\log^+ |g_j(re^{i\varphi})| \leq \frac{Cr}{\sin\varphi} \log \frac{Cr}{\sin\varphi}, \; re^{i\varphi} \in \mathbb{C}_+, \; r \geq 1, \; j = 1, \cdots, n, \tag{6.2}$$

where C is a positive constant.

Map $\mathbb{C}_+$ onto $D = \{\zeta \in \mathbb{C} : \; |\zeta| < 1\}$ by

$$\zeta = \frac{z-i}{z+i} \tag{6.3}$$

and set

$$\tilde{h}(\zeta) = h(z), \; \tilde{g}_j(\zeta) = g_j(z), \; j = 1, \cdots, n.$$

Then

$$\begin{aligned} \tilde{h}(\zeta) &= \tilde{g}_1(\zeta) \cdots \tilde{g}_n(\zeta), \\ \tilde{g}_n(\zeta) &= \tilde{g}_1(\zeta) + \cdots + \tilde{g}_{n-1}(\zeta). \end{aligned} \tag{6.4}$$

Moreover, $\tilde{h}$ is bounded in D and $\tilde{g}_1, \cdots \tilde{g}_{n-1}$ are linearly independent over $\mathbb{C}$. Since the zeros of h satisfy the Blaschke condition for the upper half-plane, the zeros of $\tilde{h}$ satisfy the Blaschke condition for the unit disk, equation (6.4) implies that the zeros of $\tilde{g}_1, \cdots \tilde{g}_{n-1}$ also satisfy this condition. Applying Theorem R (see, p.28), we get

$$T(r, \tilde{g}_j/\tilde{g}_n) = O\left(\log \frac{1}{1-r}\right), \; r \to 1, \; j = 1, \cdots, n-1. \tag{6.5}$$

Using the properties of the Nevanlinna characteristic (see, p.27) and taking into account that boundedness of $\tilde{h}$ in D implies $T(r, \tilde{h}) = O(1), \; r \to 1$, we obtain

$$\begin{aligned} T(r, \tilde{g}_n) &= T\left(r, \frac{1}{\tilde{g}_n}\right) + O(1) = \frac{1}{n} T\left(r, \frac{1}{(\tilde{g}_n)^n}\right) + O(1) \\ &\leq \frac{1}{n} T\left(r, \frac{\tilde{h}}{(\tilde{g}_n)^n}\right) + \frac{1}{n} T(r, \tilde{h}) + O(1) = \frac{1}{n} T\left(r, \frac{\tilde{h}}{(\tilde{g}_n)^n}\right) + O(1), \; r \to 1, \end{aligned}$$

whence, using (6.4) and (6.5), we conclude

$$\begin{aligned} T(r,\tilde g_n) &\le \frac{1}{n}T\left(r,\frac{\tilde g_1\cdots\tilde g_n}{(\tilde g_n)^n}\right)+O(1)\\ &\le \frac{1}{n}\sum_{j=1}^{n-1}T\left(r,\frac{\tilde g_j}{\tilde g_n}\right)+O(1)=O\left(\log\frac{1}{1-r}\right),\quad r\to 1. \end{aligned}$$

Using (6.5) once again, we get

$$T(r,\tilde g_j)\le T\left(r,\frac{\tilde g_j}{\tilde g_n}\right)+T(r,\tilde g_n)+O(1)=O\left(\log\frac{1}{1-r}\right), r\to 1, j=1,\cdots,n. \tag{6.6}$$

The inequality

$$\log^+ M(r,\tilde g_j)\le \frac{4}{1-r}T\left(\frac{1+r}{2},\tilde g_j\right)$$

allows us to derive from (6.6) that there is a positive constant C_1, such that

$$\log^+|\tilde g_j(\zeta)|\le \frac{C_1}{1-|\zeta|}\log\frac{C_1}{1-|\zeta|},\ \zeta\in D,\ j=1,\cdots,n.$$

Since (6.3) implies

$$1-|\zeta|\ge\frac{1}{2}(1-|\zeta|^2)=\frac{1}{2}\frac{|z+i|^2-|z-i|^2}{|z+i|^2}=\frac{2\operatorname{Im} z}{|z+i|^2}\ge\frac{\operatorname{Im} z}{2|z|^2},\ |z|\ge 1,$$

and $\tilde g_j(\zeta)=g_j(z)$, we get (6.2).

2. Let us show that

$$\sup\left\{\int_{-\infty}^{\infty}\frac{|\log|g_j(x+iy)||}{1+x^2}dx:\ 0<y<H\right\}<\infty,\ j=1,\cdots,n, \tag{6.7}$$

where H is taken from the condition (2.16) of Theorem 2.8.

To prove (6.7), note that

$$\begin{aligned} \int_{-\infty}^{\infty}\frac{|\log|g_j(x+iy)||}{1+x^2}dx &= \int_{-\infty}^{\infty}\frac{\log^+|g_j(x+iy)|}{1+x^2}dx\\ &\quad+\int_{-\infty}^{\infty}\frac{\log^+|1/g_j(x+iy)|}{1+x^2}dx. \end{aligned} \tag{6.8}$$

The first integral in the right hand side is bounded for $0 < y < H$ by the condition (2.16) of Theorem 2.8. Using condition (2.16) once again, we get

$$\frac{1}{|g_j(x+iy)|} = \frac{1}{|h(x+iy)|}\prod_{k\neq j}|g_k(x+iy)| \leq \frac{C}{|h(x+iy)|}, \quad 0 < y < H,$$

where C is a positive constant. Hence, the boundedness of the second integral in the right hand side of (6.8) is a consequence of Theorem L with $Q = h$ (see, p.25).

3. Denote by B_j the Blaschke product formed by the zeros of g_j and set

$$u_j := \log\left|\frac{g_j}{B_j}\right|, \quad j = 1, \cdots, n.$$

This is a harmonic function in $\mathbb{C}_+$. Let us show that it satisfies conditions of Corollary 2.2.

We have

$$\int_0^\pi u_j^+(re^{i\varphi})\sin\varphi d\varphi \leq \int_0^\pi \log^+|g_j(re^{i\varphi})|\sin\varphi d\varphi + \int_0^\pi \log^+\frac{1}{|B_j(re^{i\varphi})|}\sin\varphi d\varphi. \quad (6.9)$$

The estimate (6.2) implies that the first integral in the right hand side is $O(r\log r)$ as $r \to \infty$. Applying Lemma 4.1 to $B_j(z)$, we get that the second integral in (6.9) is $O(r)$ as $r \to \infty$. Hence, u_j satisfies the condition (2.7) of Corollary 2.2.

Further we have

$$\int_{-\infty}^\infty \frac{|u_j(x+iy)|}{1+x^2}dx \leq \int_{-\infty}^\infty \frac{|\log|g_j(x+iy)||}{1+x^2}dx + \int_{-\infty}^\infty \frac{\log^+|1/B_j(x+iy)|}{1+x^2}dx.$$

Using (6.7) for the first integral in the right hand side and Theorem L for the second one, we see that condition (2.8) of Corollary 2.2 is also satisfied.

4. Let us complete the proof of Theorem 2.8.

Applying Corollary 2.2, we get the representation

$$u_j(z) = \frac{y}{\pi}\int_{-\infty}^\infty \frac{d\nu_j(t)}{(t-x)^2+y^2} + k_j y, \quad z = x+iy \in \mathbb{C}_+,$$

where k_j is a real constant, ν_j is a real-valued Borel measure such that

$$\int_{-\infty}^{\infty} \frac{d|\nu_j|(t)}{1+t^2} < \infty.$$

Thus,

$$\log|g_j(z)| = \log|B_j(z)| + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\nu_j(t)}{(t-x)^2+y^2} + k_j y, \quad z = x+iy \in \mathbb{C}_+,$$

and hence, g_j belongs to the Nevanlinna class. Using Lemma 4.4, we conclude that $g_j(z)e^{i\alpha_j z} \in \mathbb{C}_+$ for some real number α_j. □

6.3 A criterion of belonging to $H^p(\mathbb{C}_+)$ up to an exponential factor

Proof of Theorem 2.9.

Case I. Firstly, we prove the theorem under additional assumption that f is analytic in the closed half-plane $\overline{\mathbb{C}_+}$ and does not vanish on $\mathbb{R}$.

Let B be the Blaschke product formed by the zeros of f. Set

$$g := \frac{f}{B}, \quad u := \log|g|. \tag{6.10}$$

The function g is analytic and non-vanishing in the closed half-plane $\overline{\mathbb{C}_+}$ and therefore u is harmonic in $\overline{\mathbb{C}_+}$. We will show that u satisfies the conditions of Corollary 2.5 with $\alpha = 2$.

We have

$$\int_0^{\pi} u^+(re^{i\varphi})\sin\varphi d\varphi \le \int_0^{\pi} \log^+|f(re^{i\varphi})|\sin\varphi d\varphi + \int_0^{\pi} \log^+\frac{1}{|B(re^{i\varphi})|}\sin\varphi d\varphi.$$

The second integral in the right hand side is $O(r)$, as $r \to \infty$ (see, Lemma 4.1, p.30). Therefore condition (2.5) follows from (2.18).

Further,

$$\int_{-\infty}^{\infty} \frac{|u(x+iy)|}{1+x^2}dx \le \int_{-\infty}^{\infty} \frac{|\log|f(x+iy)||}{1+x^2}dx + \int_{-\infty}^{\infty} \frac{\log^+|1/B(x+iy)|}{1+x^2}dx.$$

The second integral in the right hand side is bounded on any finite interval of values of $y > 0$ (see, Theorem L, p.25). For the first integral, we have

$$\int_{-\infty}^{\infty} \frac{|\log |f(x+iy)||}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{\log^+ |f(x+iy)|}{1+x^2} dx + \int_{-\infty}^{\infty} \frac{\log^- |f(x+iy)|}{1+x^2} dx$$
$$\leq C_p \int_{-\infty}^{\infty} |f(x+iy)|^p dx + \int_{-\infty}^{\infty} \frac{\log^- |f(x+iy)|}{1+x^2} dx.$$

Therefore, the condition (2.12) follows from (2.19) and (2.20).

So, Corollary 2.5 is applicable to function u defined by (6.10), and hence the representation (2.9) holds with $\nu(t) = \log |g(t)| dt$.

Since $|B(t)| = 1$ for $t \in \mathbb{R}$, we have

$$|g(t)| = |f(t)|, \quad d\nu(t) = \log |f(t)| dt.$$

Hence, the representation (2.9) can be rewritten in the form

$$\log |g(z)| = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log |f(t)|}{(t-x)^2 + y^2} dt + ky, \quad z = x + iy \in \mathbb{C}_+.$$

Taking into account that $|f(z)| \leq |g(z)|,\ z \in \mathbb{C}_+$, we obtain

$$\log |f(z)| \leq \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log |f(t)|}{(x-t)^2 + y^2} dt + ky.$$

It follows from the above representation and arithmetic-geometric mean inequality (see, e.g. [7, p.29]) that

$$|f(z)e^{ikz}|^p \leq \exp\left\{ \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log(|f(t)|^p)}{(x-t)^2 + y^2} dt \right\}$$
$$\leq \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{|f(t)|^p}{(x-t)^2 + y^2} dt, \quad z = x + iy \in \mathbb{C}_+.$$

By Fubini's theorem, we have

$$\int_{-\infty}^{\infty} |f(x+iy)e^{ik(x+iy)}|^p dx \leq \int_{-\infty}^{\infty} |f(t)|^p \left\{ \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{1}{(x-t)^2 + y^2} dx \right\} dt$$
$$= \int_{-\infty}^{\infty} |f(t)|^p dt,$$

and hence $f(z)e^{ikz} \in H_p(\mathbb{C}_+)$.

Case II. Now, let f be any function satisfying conditions (2.17)-(2.20).

Let $s \in (0, H)$ be such that $f(t + is) \neq 0$ for $t \in \mathbb{R}$. Set

$$f_s(z) := f(z + is).$$

This function is analytic in the closed half-plane $\overline{\mathbb{C}_+}$ and does not vanish on $\mathbb{R}$. It suffices to show that

$$\|f(\cdot + iy)e^{ik(\cdot+iy)}\|_p \tag{6.11}$$

is bounded for $s \leq y < \infty$ (for some $k \in \mathbb{R}$). Boundedness of (6.11) for $0 < y < s$ follows from (2.20) because $s < H$.

In order to apply the result of part I of the proof, we should check that conditions (2.17), (2.18), (2.19), (2.20) are satisfied for $f_s(z)$. It is trivial that (2.19) and (2.20) are satisfied with $H - s$ instead of H because $0 < s < H$.

To check (2.17), note that, if z_k is a zero of f, then $z_k - is$ is a zero of f_s. Therefore,

$$\sum_{\operatorname{Im} z_k > s} \frac{\operatorname{Im}(z_k - is)}{1 + |z_k - is|^2} \leq \left(\max_{\operatorname{Im} z_k > s} \frac{1 + |z_k|^2}{1 + |z_k - is|^2} \right) \sum_{\operatorname{Im} z_k > s} \frac{\operatorname{Im} z_k}{1 + |z_k|^2} < \infty.$$

It remains to check (2.18).

Let

$$Q_{R,s} := \{z \in \mathbb{C} : \; |z + is| < R\} \cap \mathbb{C}_+, \quad R > s.$$

Note that the function $\log^+ |f_s(z)|$ is subharmonic in the closure of $Q_{R,s}$. Therefore, using the arguments in pages 51-51, we obtain

$$\begin{aligned} \log^+ |f_s(re^{i\varphi})| \leq \frac{2(R + r + s)^3}{\pi(R - r - s)^3} \int_0^\pi \log^+ |f(Re^{i\theta})| \sin\theta d\theta \\ + \frac{1}{\pi} \int_{-\infty}^{\infty} \log^+ |f(t + is)| \frac{r \sin\varphi dt}{r^2 + t^2 - 2rt\cos\varphi}, \quad re^{i\varphi} \in Q_{R,s}. \end{aligned} \tag{6.12}$$

For $0 < r < R - s$, we have $re^{i\varphi} \in Q_{R,s}$ for $0 < \varphi < \pi$. Let us multiply both sides of (6.12) by $\sin\varphi$ and integrate with respect to φ from 0 to π. Using the relation (4.12), we obtain

$$\int_0^\pi \log^+ |f_s(re^{i\varphi})| \sin\varphi d\varphi \le \frac{4(R+r+s)^3}{\pi(R-r-s)^3} \int_0^\pi \log^+ |f(Re^{i\theta})| \sin\theta d\theta + r\int_{-\infty}^{\infty} \log^+ |f(t+is)| \frac{dt}{1+t^2}. \tag{6.13}$$

Let $\{r_k\}$ be the sequence staying in (2.18). Put $R = r_k,\ r = r_k/2 - s$ in (6.13). Then we see that condition (2.18) is satisfied for f_s with $\{r_k/2 - s\}$ instead of $\{r_k\}$. □

Now, let us consider examples showing that conditions (2.17), (2.18), (2.19), (2.20) are independent, and moreover, (2.18) and (2.20) cannot be substantially weakened.

Example 6.4. Let

$$E_\rho(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k/\rho)}, \quad \rho > 1,$$

be Mittag-Leffler's entire function. It is known (see, e.g., [3, p.275]) that the following asymptotic formula holds as $z \to \infty$:

$$E_\rho(z) = \begin{cases} -\frac{1}{z\Gamma(1-1/\rho)} + O(\frac{1}{|z|^2}), & \frac{\pi}{2\rho} \le \arg z \le 2\pi - \frac{\pi}{2\rho}, \\ \rho e^{z^\rho} + O(\frac{1}{|z|}), & -\frac{\pi}{2\rho} \le \arg z \le \frac{\pi}{2\rho}. \end{cases}$$

This implies that the function

$$f(z) = (z+i)^{-2/p} E_\rho(-iz)$$

satisfies (2.18), (2.19), (2.20).

Since each $H_p(\mathbb{C}_+)$ is a subclass of the Nevanlinna class, it follows from Theorem K, p.24 that if a function belongs to $H_p(\mathbb{C}_+)$ up to factor $e^{ikz},\ k \in \mathbb{R}$, it should belong to the Nevanlinna class.

Therefore, f does not belong to $H_p(\mathbb{C}_+)$ up to an exponential factor because

$$f(iy) = \rho(y+1)^{-2/p} e^{y^\rho}(1+o(1)),\ y \to \infty,$$

and this contradicts to the fact that any function belonging to the Nevanlinna class should satisfy the estimate (6.1).

Example 6.5. The function

$$f(z) = e^{-z^2}$$

satisfies (2.17), (2.18), (2.20). But it does not belong to $H_p(\mathbb{C}_+)$ up to an exponential factor (see, Example 6.3). Here (2.19) is violated.

Example 6.6. The function

$$f(z) = (z+i)^{-2/p} e^{-iz^3}$$

satisfies (2.17), (2.18), (2.19), but f does not belong to $H_p(\mathbb{C}_+)$ up to an exponential factor. Here (2.20) is violated. This example also shows that Theorem 2.9 ceases to be true if (2.20) is replaced with $\|f(\cdot + i0)\|_p < \infty$.

Example 6.7. The function

$$f(z) = (z+i)^{-2/p} \exp\exp(-ciz), \quad c > 0,$$

satisfies (2.17), (2.19), (2.20), but evidently it does not belong to $H_p(\mathbb{C}_+)$ up to an exponential factor (e.g., f does not satisfy (6.1)). Here (2.18) is violated.This example also shows that Theorem 2.9 ceases to be true if "O" is replaced with "o" in (2.18).

Chapter 7

Application to generalization of the Titchmarsh convolution theorem

Proof of Theorem 2.10. If $\ell(\mu_j) > -\infty,\ j = 1, \cdots, n$, then (2.26) follows from the Titchmarsh theorem (see, Theorem S, p.29). If $\ell(\mu_1 * \cdots * \mu_n) = -\infty$, then (2.26) follows from the inequality (3.16). Hence, Theorem 2.10 will be proved if we prove the following fact.

Theorem 7.1 *Under the hypotheses of Theorem 2.10, the following implication holds:*

$$\ell(\mu_1 * \mu_2 * \cdots * \mu_n) > -\infty \Rightarrow \ell(\mu_j) > -\infty,\ j = 1, \ldots, n.$$

Proof. Let $\mu := \mu_1 * \mu_2 * \cdots * \mu_n$. Without loss of generality we can assume that $\ell(\mu_1 * \mu_2 * \cdots * \mu_n) = 0$. Thus, the Fourier transform $\hat{\mu}$ of the measure μ satisfies

$$|\hat{\mu}(z)| = \left| \int_0^\infty e^{itz} d\mu(t) \right| \leq \int_0^\infty e^{-yt} d|\mu|(t) \leq |\mu|((0, \infty)), \quad z = x + iy \in \overline{\mathbb{C}_+}.$$

Therefore $\hat{\mu}$ belongs to $H^\infty(\mathbb{C}_+)$ and continuous on $\overline{\mathbb{C}_+}$.

For $z \in \overline{\mathbb{C}_+}$ we have

$$\begin{aligned}\hat{\mu}_j(z) &= \int_{-\infty}^{\infty} e^{itz} d\mu_{j,1}(t) + \int_{-\infty}^{\infty} e^{itz} d\mu_{j,2}(t) \\ &=: \hat{\mu}_{j,1}(z) + \hat{\mu}_{j,2}(z),\end{aligned}$$

where $\mu_{j,1}$ and $\mu_{j,2}$ are the restrictions of μ_j to $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively. Similar to $\hat{\mu}$, $\hat{\mu}_{j,2}$ belongs to $H^\infty(\mathbb{C}_+)$ and continuous on $\overline{\mathbb{C}_+}$ and $\hat{\mu}_{j,1}$ is bounded analytic function on $\mathbb{C}_- := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ and continuous on $\overline{\mathbb{C}_-} := \{z \in \mathbb{C} : \operatorname{Im} z \leq 0\}$. We will show that $\hat{\mu}_{j,1}$ converges absolutely and uniformly on any horizontal strip of $\mathbb{C}_+$ and therefore represents an entire function.

Note that

$$\begin{aligned}|\hat{\mu}_{j,1}(z)| &\leq \int_{-\infty}^{0} e^{-yt} d|\mu_j|(t) \\ &= e^{-yt}|\mu_j|((-\infty, t))\Big|_{-\infty}^{0} + y\int_{-\infty}^{0} e^{-yt}|\mu_j|((-\infty, t))dx.\end{aligned}$$

Fix $c > 0$. It follows from the condition (2.25) that there exists $N_c > 0$ and $A_c > 0$ such that $|\mu_j|((-\infty, t)) \leq A_c e^{-ct}$, $t \leq -N_c$. For $0 \leq y < c$, we have

$$|\hat{\mu}_j(z)| \leq |\mu_j|((-\infty, 0)) + cA_c \int_{-\infty}^{-N_c} e^{(c-y)t} dt + c\int_{-N_c}^{0} e^{-yt}|\mu_j|((-\infty, 0))dt = B_c.$$

Thus, $\hat{\mu}_j$ is a function analytic in $\mathbb{C}_+$ and uniformly bounded in each horizontal strip of $\mathbb{C}_+$. For $z \in \mathbb{R}$, we have

$$\hat{\mu} = \hat{\mu}_1 \hat{\mu}_2 \cdots \hat{\mu}_n. \tag{7.1}$$

Since, $\hat{\mu}$ and $\hat{\mu}_j$, $j = 1, \cdots, n$ are continuous on $\overline{\mathbb{C}_+}$, the equation (7.1) holds in $\overline{\mathbb{C}_+}$.

Taking into account that $\mu_n = \mu_1 + \cdots + \mu_{n-1}$, we see that Theorem 2.8 is applicable to $h = \hat{\mu}$, $g_j = \hat{\mu}_j$, $j = 1, \cdots, n$. By Theorem 2.8, we obtain that $\hat{\mu}_j(z)\exp\{ib_j z\} \in H^\infty(\mathbb{C}_+)$, $j = 1, \cdots, n$. Thus,

$$|\hat{\mu}_{j,1}(z)| \leq C_1 e^{|b_j||z|}, \ z \in \mathbb{C}_+.$$

We have already noted that $\hat{\mu}_{j,1}$ is bounded on $\overline{\mathbb{C}_-}$. Using the well-known corollary of the Paley-Wiener theorem (see, Theorem T, p.29), we get $\operatorname{supp} \mu_{j,1} \subset$

$[-|b_j|, |b_j|]$ and hence $\operatorname{supp} \mu_j \subset [-|b_j|, \infty)$. This proves that $\ell(\mu_j) > -\infty$, $j = 1, \cdots, n$. $\square$

The following example shows that the condition (2.25) in Theorem 2.10 cannot be weakened by replacing '$\forall$' by '$\exists$'.

Example 7.1. Let μ_1, μ_2, μ_3 be the measures defined by the Fourier transforms

$$\hat{\mu}_1(z) = \frac{1}{1 + iz/c}, \ \hat{\mu}_2(z) = \frac{(1 + iz/c)^2}{(1 - iz/c)^4}, \ \hat{\mu}_3(z) = \hat{\mu}_1(z) + \hat{\mu}_2(z).$$

where c is a positive constant. The direct calculation of the inverse Fourier transform shows that the condition (2.25) is satisfied with the given fixed $c > 0$, and $\ell(\mu_1) = \ell(\mu_2) = \ell(\mu_3) = -\infty$. Nevertheless, the product $\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3$ belongs to $H^2(\mathbb{C}_+) \cap H^\infty(\mathbb{C}_+)$ and therefore $\ell(\mu_1 * \mu_2 * \mu_3) = 0$ (see, [2, p.75]).

Bibliography

[1] L. V. Ahlfors, *Complex analysis: An introduction to the theory of analytic functions of one complex variable*, third ed., McGraw-Hill Book Co., New York, 1978.

[2] N. I. Akhiezer, *Lectures on integral transforms*, Translations of Mathematical Monographs, vol. 70, American Mathematical Society, Providence, RI, 1988.

[3] L. Bieberbach, *Lehrbuch der Funktionentheorie. I. Elemente der Funktionentheorie. II. Moderne Funktionentheorie*, Chelsea Publishing Company, New York, 1945.

[4] T. Carleman, *Sur les fonctions inverses des fonctions entières d'ordre fini*, Ark. Mat. **15** (1921), no. 10, 1–7.

[5] H. Cartan, *Sur les zéros des combinaisons linéaires de p fonctions holomorphes donnes*, Mathematica, Cluji **7** (1933), 5–31.

[6] Y. Domar, *Extensions of the Titchmarsh convolution theorem with applications in the theory of invariant subspaces*, Proc. London Math. Soc. (3) **46** (1983), no. 2, 288–300.

[7] P. L. Duren, *Theory of H^p spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York, 1970.

[8] S. Gergün and I. V. Ostrovskii, *On the Poisson representation of a function harmonic in the upper half-plane*, Mat. Fiz. Anal. Geom. **9** (2002), no. 2, 243–248.

[9] ______, *On the Poisson representation of a function harmonic in the upper half-plane*, Comput. Methods Func. Theory **2** (2002), no. 1, 191–213.

[10] ______, *On a criterion of belonging to the Hardy class $H_p(\mathbb{C}_+)$ up to an exponential factor*, Mat. Fiz. Anal. Geom. (to appear).

[11] S. Gergün, I. V. Ostrovskii, and A. Ulanovskii, *On the Titchmarsh convolution theorem*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 1, 41–46.

[12] ______, *On the Titchmarsh convolution theorem*, Ark. Mat. **40** (2002), no. 1, 55–71.

[13] A. A. Gol′dberg and I. V. Ostrovskiĭ, *Raspredelenie znachenii meromorfnykh funktsii*, Izdat. "Nauka", Moscow, 1970.

[14] N. V. Govorov, *Riemann's boundary problem with infinite index*, Operator Theory: Advances and Applications, vol. 67, Birkhäuser Verlag, Basel, 1994.

[15] K. Hoffman, *Banach spaces of analytic functions*, Dover Publications Inc., New York, 1988.

[16] P. Koosis, *Introduction to H_p spaces*, second ed., Cambridge Tracts in Mathematics, vol. 115, Cambridge University Press, Cambridge, 1998.

[17] ______, *The logarithmic integral. I*, Cambridge Studies in Advanced Mathematics, vol. 12, Cambridge University Press, Cambridge, 1998.

[18] B. Ya. Levin, *Distribution of zeros of entire functions*, revised ed., Translations of Mathematical Monographs, vol. 5, American Mathematical Society, Providence, R.I., 1980.

[19] ______, *Lectures on entire functions*, Translations of Mathematical Monographs, vol. 150, American Mathematical Society, Providence, RI, 1996.

[20] Ju. V. Linnik and Ĭ. V. Ostrovs′kiĭ, *Decomposition of random variables and vectors*, American Mathematical Society, Providence, R. I., 1977.

[21] R. Nevanlinna, *Über die Eigenschaften meromorpher Funktionen in einem Winkelraum*, Acta Soc. Sci. Fenn. **50** (1925), no. 12, 1–45.

[22] ———, *Analytic functions*, Springer-Verlag, New York, 1970.

[23] I. V. Ostrovskiĭ, *Generalization of the Titchmarsh convolution theorem and the complex-valued measures uniquely determined by their restrictions to a half-line*, Stability problems for stochastic models (Uzhgorod, 1984), Lecture Notes in Math., vol. 1155, Springer, Berlin, 1985, pp. 256–283.

[24] E. C. Titchmarsh, *The zeros of certain integral functions*, Proc. London Math. Soc. **25** (1926), 283–302.